微分積分学第一 (LAS.M101-06)

チェイン・ルール

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$z =$

- $f(x, y)$：微分可能な 2 変数関数
- $x = x(\xi, \eta)$, $y = y(\xi, \eta)$：微分可能 ✓
- $z = f(\xi, \eta) := f\big(x(\xi, \eta), y(\xi, \eta)\big)$

$\Rightarrow$

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$
$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}.$$

変換 $(\xi, \eta) \mapsto (x, y)$ の Jacobi行列

$$\left(\frac{\partial f}{\partial \xi}, \frac{\partial f}{\partial \eta}\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix}$$

変数変換

主張 (命題 4.9)

平面の領域 D から U への 1 対 1 上への C^1-級写像

$$F\colon D \ni (\xi,\eta) \longmapsto (x,y) \in U$$

を考える．とくにその逆写像

$$G = F^{-1}\colon U \ni (x,y) \longmapsto (\xi,\eta) \in D$$

も C^1-級ならば，F の微分（ヤコビ行列）は正則で，

(仮定)

$$\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix}^{-1}$$

が成り立つ．

$\cdot\ F : (\xi, \eta) \longmapsto (x(\xi,\eta), y(\xi,\eta))$

$\cdot\ G = F^{-1} : (x, y) \longmapsto (\xi(x,y), \eta(x,y))$

① $\xi(x(\xi,\eta), y(\xi,\eta)) = \xi$

② $\eta(x(\xi,\eta), y(\xi,\eta)) = \eta$

$\frac{\partial ①}{\partial \xi} : x_\xi \xi_x + y_\xi \xi_y = 1 \qquad \frac{\partial ②}{\partial \xi} \qquad 0$

$\frac{\partial ①}{\partial \eta} : x_\eta \xi_x + y_\eta \xi_y = 0 \qquad \frac{\partial ②}{\partial \eta} \qquad 1$

$$\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

例題：波動方程式と d'Alembert の解

$$\begin{cases} \xi = x + ct \\ \eta = x - ct \end{cases}$$

逆変換

$$\begin{pmatrix} \xi_x & \xi_t \\ \eta_x & \eta_t \end{pmatrix} = \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix}$$

c：定数； $(c \neq 0)$

C^2-級の 2 変数関数 $f(t, x)$ に対して

$$Df := \frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2}$$

変数変換：

$$x = \frac{1}{2}(\xi + \eta), \qquad \overset{t}{\cancel{y}} = \frac{1}{2c}(\xi - \eta)$$

$$\begin{pmatrix} x_\xi & x_\eta \\ t_\xi & t_\eta \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2c} & -\frac{1}{2c} \end{pmatrix}$$

$\xi = x + ct$

$\eta = x - ct$

- $f(x, t)$
- $\rightsquigarrow f(x(\xi, \eta), t(\xi, \eta))$

$$\frac{\partial f}{\partial x} = \xi_x \frac{\partial f}{\partial \xi} + \eta_x \frac{\partial f}{\partial \eta} = \underbrace{\frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \eta}}_{(*)}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \frac{\partial f}{\partial x} = \xi_x \frac{\partial}{\partial \xi}\left((*) \right) + \eta_x \frac{\partial}{\partial \eta}\left((*) \right)$$

$$= \frac{\partial}{\partial \xi}\left(\frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \eta} \right) + \frac{\partial}{\partial \eta}\left(\frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \eta} \right)$$

$$= \frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \xi \partial \eta} + \frac{\partial^2 f}{\partial \eta \partial \xi} + \frac{\partial^2 f}{\partial \eta^2}$$

$$\frac{\partial^2 f}{\partial \xi \partial \eta} = \frac{\partial^2 f}{\partial \eta \partial \xi}$$

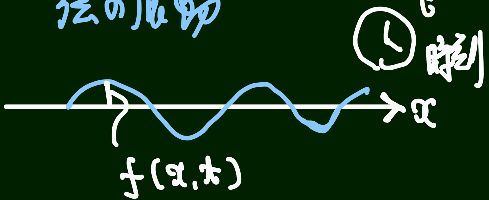
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial \xi^2} + 2\frac{\partial^2 f}{\partial \xi \partial \eta} + \frac{\partial^2 f}{\partial \eta^2}$$

$$\frac{\partial f}{\partial t} = \xi_t \frac{\partial f}{\partial \xi} + \eta_t \frac{\partial f}{\partial \eta} = c\left(\frac{\partial f}{\partial \xi} - \frac{\partial f}{\partial \eta}\right)$$

$$\frac{\partial^2 f}{\partial t^2} = c^2\left(\frac{\partial^2 f}{\partial \xi^2} - 2\frac{\partial^2 f}{\partial \xi \partial \eta} + \frac{\partial^2 f}{\partial \eta^2}\right)$$

$$\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = 0$$ ：波動方程式

弦の振動



$$\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = 0 \quad \circledast \qquad \xi = x + ct,\quad \eta = x - ct$$

$$\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = -4c^2 \frac{\partial^2 f}{\partial \xi \partial \eta}.$$

$$\circledast \iff \frac{\partial^2 f}{\partial \xi \partial \eta} = 0$$

d'Alembertの解.

e.g. $\sin(x+ct) + e^{x-ct}$

$$\Rightarrow \frac{\partial}{\partial \xi} \frac{\partial f}{\partial \eta} = 0 \Rightarrow$$ $\frac{\partial f}{\partial \eta}$ は η のみの関数

$$\Rightarrow f = (\eta\text{のみの関数}) + (\xi\text{のみの関数})$$

$$= F(\xi) + G(\eta) = F(x+ct) + G(x-ct)$$

例題：ラプラシアンと調和関数

$r = \sqrt{x^2+y^2}$

C^2-級の 2 変数関数 $f(x,y)$ に対して

$$\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

$\frac{\partial r}{\partial x} = \cos\theta$

$\cancel{\frac{\partial x}{\partial r}} = \frac{x}{\sqrt{x^2+y^2}}$

変数変換

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$(r, \theta) \longmapsto (x, y)$ 極座標

$$\begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta, & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \cos\theta \end{pmatrix}$$

$r = r(x, y)$

$\theta = \theta(x, y)$

$$\frac{\partial}{\partial x} = r_x \frac{\partial}{\partial r} + \theta_x \frac{\partial}{\partial \theta} = \cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\Big(\underbrace{\cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}}_{}\Big) \quad \circledast$$

$$= \cos\theta \frac{\partial}{\partial r} \circledast - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta} \circledast$$

$$= \cos^2\theta \frac{\partial^2}{\partial r^2} - \cos\theta \left\{ -\frac{1}{r^2}\sin\theta \frac{\partial}{\partial \theta} + \frac{1}{r}\sin\theta \frac{\partial^2}{\partial r \partial \theta} \right\}$$

$$+ -\frac{1}{r}\sin\theta \left(-\sin\theta \frac{\partial}{\partial r} + \cos\theta \frac{\partial^2}{\partial \theta \partial r} \right)$$

$$+ \frac{1}{r^2}\sin\theta\cos\theta \frac{\partial}{\partial \theta} + \frac{1}{r^2}\sin^2\theta \frac{\partial^2}{\partial \theta^2}$$

$$\frac{\partial^2}{\partial x^2} = \cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{1}{r}\sin^2\theta \frac{\partial}{\partial r} + \frac{1}{r^2}\sin^2\theta \frac{\partial^2}{\partial \theta^2}$$

$$- 2\cos\theta \sin\theta \frac{\partial^2}{\partial r \partial \theta} + \frac{2}{r^2}\cos\theta \sin\theta \frac{\partial}{\partial \theta}$$

etc.

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \underset{\text{Laplacian}}{\Delta} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Laplacian の
极座标表示