Advanced Topics in Geometry A1 (MTH.B405)

Overview

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Our Goal

Theorem (The Fundamental Theorem for surface theory)

Let

- $U \subset \mathbb{R}^2$ *be a simply connected domain,*
- I *be a positive definite symmetric quadratic form on* U
- II *be a symmetric quadratic form on* U.
- I *and* II *satisfy the Gauss and Codazzi equations.*

$\Rightarrow$

- *there exists a surface* $\boldsymbol{f} : U \to \mathbb{R}^3$ *with first and second fundamental forms* I *and* II*, resp.*
- *such an* $\boldsymbol{f}$ *is unique up to orientation preserving isometry of* $\mathbb{R}^3$*.*

Surfaces

A surface in $\mathbb{R}^3$:

Surfaces

Olympiastadion München (1971); Frei Otto

Expression of Surfaces—Parametrization

- $U \subset \mathbb{R}^2$: a domain
- $\boldsymbol{f} : U \longrightarrow \mathbb{R}^3$

Parametrization—Example

$$\boldsymbol{f}\colon\ (-\pi,\pi)\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\ni(u,v)\longmapsto\boldsymbol{f}(u,v)=\begin{pmatrix}\cos u\cos v\\ \sin u\cos v\\ \sin v\end{pmatrix}\in\mathbb{R}^3$$

$\xrightarrow{\ }$ $\boldsymbol{f}$

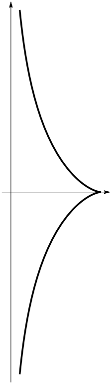
Parametrization—Example

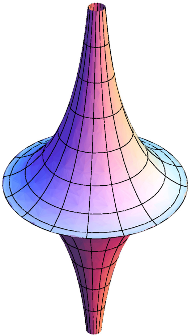
$$\boldsymbol{g}\colon\ (-\pi,\pi)\times(-\infty,\infty)\ni(u,v)\longmapsto\boldsymbol{g}(s,t)=\begin{pmatrix}\cos s\,\operatorname{sech}t\\ \sin s\,\operatorname{sech}t\\ \tanh t\end{pmatrix}\in\mathbb{R}^3$$

$$\underset{\boldsymbol{g}}{\longrightarrow}$$

Parametrization—Example

$$\boldsymbol{h}\colon\ (-\pi,\pi)\times(-\infty,\infty)\ni(u,v)\longmapsto\boldsymbol{h}(s,t)=\begin{pmatrix}\cos s\,\operatorname{sech} t\\ \sin s\,\operatorname{sech} t\\ t-\tanh t\end{pmatrix}\in\mathbb{R}^3$$





A Naive Question

Q

What quantities determine the surface?

- a triple of three coordinate functions of $\boldsymbol{f}(u, v)$.

Isometries:

$$\boldsymbol{f} \longmapsto A\boldsymbol{f} + \boldsymbol{a}, \qquad A \in \mathrm{SO}(3), \boldsymbol{a} \in \mathbb{R}^3$$

A Naive Question

Q

What quantities determine the surface up to “congruence”?

A

The first and second fundamental forms.

The Fundamental Theorem

Theorem (The Fundamental Theorem for surface theory)

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- I *and* II *satisfy the Gauss and Codazzi equations.*

$\Rightarrow$

- $\exists$*a surface* $\boldsymbol{f} : U \to \mathbb{R}^3$ *with* I *and* II,
- $\boldsymbol{f}$ *is unique up to orientation preserving isometry.*