Advanced Topics in Geometry A1 (MTH.B405)

Ordinary Differential Equations

Kotaro Yamada

kotaro@math.sci.isct.ac.jp

http://www.official.kotaroy.com/class/2025/geom-a1

Institute of Science Tokyo

2025/04/25

Exercise 1-1

Problem (Ex. 1-1)

Let $f(x,y) = e^{ax}\cos y$*, where* a *is a constant. Find a function* $g(x,y)$ *satisfying*

$$g_x = -f_y, \quad g_y = f_x, \quad g(0,0) = 0.$$

By commutativity if such g exists

then $g_{xy} = -f_{yy}$ and $g_{yx} = f_{xx}$

must coincide. i.e.

$$f_{xx} + f_{yy} = 0 \leftarrow$$

necessary condition for existence of g

$$\Leftrightarrow \quad a = \pm 1$$

$$f(x,y) = e^{ax}\cos y, \qquad g_x = -f_y, \qquad g_y = f_x, \qquad g(0,0) = 0.$$

When $a = 1$, -1

- $g_x = -f_y = -e^{x}\sin y$ ($+$, e^{-x})
- $g_y = f_x = e^{x}\cos y$ (e^{-x})

$\Rightarrow$ $g(x,y) = e^{x}\sin y$ $\quad$ $g(x,y) = -e^{-x}\sin y$

- $g_x = -f_y \qquad g_y = f_x$

Cauchy-Riemann equation for

$$x+iy \longmapsto f(x,y) + i\, g(x,y)$$

$$\| $$

$\underline{a=1}$

$$e^{x}(\cos y + i \sin y)$$

$$= e^{x+iy}$$

holomorphic

$\underline{a=-1}$

$$= e^{-(x+iy)}$$

Exercise 1-2

Problem (Ex. 1-2)

α: *a* 1*-form on* $U = \mathbb{R}^2 \setminus \{(t,0)\,;\, t \leqq 0\}$ *as*

$$\alpha = a(x,y)\,dx + b(x,y)\,dy := \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

(x, y)

$\mathrm{P} = (r\cos\theta, r\sin\theta) \in U \;\; (r > 1,\, 0 < \theta < \pi),$

$$c_1(t) := \big(x_1(t), y_1(t)\big) = (\cos t, \sin t) \qquad (0 \leqq t \leqq \theta),$$
$$c_2(s) := \big(x_2(s), y_2(s)\big) = (s\cos\theta, s\sin\theta) \qquad (1 \leqq s \leqq r),$$

$\boxed{df = \alpha}$

$\shortparallel \; f(x, y)$

$$\int_{c_1 \cup c_2} \alpha := \int_0^\theta \left(a(x_1(t), y_1(t))\,\frac{dx_1}{dt}\,dt + b(x_1(t), y_1(t))\,\frac{dy_1}{dt}\,dt \right)$$
$$+ \int_1^r \left(a(x_2(s), y_2(s))\,\frac{dx_2}{ds}\,ds + b(x_2(s), y_2(s))\,\frac{dy_2}{ds}\,ds \right) = ?$$

Exercise 1-2; Differential Forms

U : a domain of the xy-plane

- $\alpha = a(x,y)\,dx + b(x,y)\,dy$: a 1-form
- $\beta = c(x,y)\,dx \underbrace{\wedge}_{\text{definition}} dy$: a 2-form

wedge

$$df := f_x\,dx + f_y\,dy \qquad f = f(x,y)\text{: a function}$$

$$d\alpha := (b_x - a_y)\,dx \wedge dy \qquad \alpha = a\,dx + b\,dy\text{: a 1-form.}$$

Commutativity of partial differentials :

$$d(df) = (f_{yx} - f_{xy})\,dx \wedge dy = 0$$

Exercise 1-2; Differential Forms

for 1-forms $\alpha\wedge\beta = -\beta\wedge\alpha$

$dx\wedge dy = -dy\wedge dx$

The Exterior Product: 外積

$(1\text{-form}) \wedge (1\text{-form}) = (2\text{-form})$; bilinear, skew-symmetric

$dx\wedge dx = 0$
$dy\wedge dy = 0$

The Exterior Derivative: of 1 form

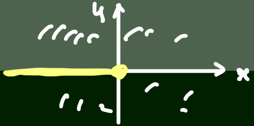
✓ $d(f\alpha) = df \wedge \alpha + f\, d\alpha$, $\quad d(dx) = d(dy) = 0$

function, 1 form $\quad$ coordinates

(f: function, α: 1-form)

$$d(a\,dx + b\,dy)$$
$$= (da)\wedge dx + a\cdot \cancel{d\,dx} + db\wedge dy + \cancel{b\, d\cdot dy}$$
$$= (a_x dx + a_y dy)\wedge dx + (b_x dx + b_y dy)\wedge dy$$
$$= a_x \cancel{dx\wedge dx} + a_y \underbrace{dy\wedge dx}^{-(dx\wedge dy)} + b_x dx\wedge dy + b_y \cancel{dy\wedge dy}$$

Exercise 1-2; Poincaré lemma



$U = \mathbb{R}^2 \setminus \{(t, 0)\,;\, t \leqq 0\}$

$(x, y) \neq (0, 0)$

$$\alpha = \frac{-y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy \qquad \Rightarrow \qquad d\alpha = 0$$

$\exists ?\ f$ such that $df = \alpha$

Theorem (Poincaré lemma)

3rd lecture

- $U \subset \mathbb{R}^2$: *a simply connected domain*
- α: *1-form on* U.
- $d\alpha = 0$

$\Rightarrow$

- $\exists f$ *such that* $df = \alpha$.

$(d\,df = 0)$

$d\alpha = 0$

$$U=\mathbb{R}^2\setminus\{(t,0)\,;\,t\leqq 0\},\ \mathrm{P}=(r\cos\theta,r\sin\theta)$$

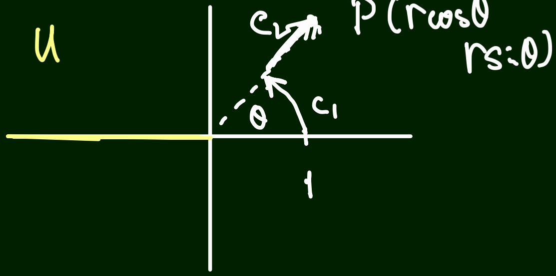
$$\alpha=\frac{-y}{x^2+y^2}\,dx+\frac{x}{x^2+y^2}\,dy$$

$$c_1(t):=\bigl(x_1(t),y_1(t)\bigr)=(\cos t,\sin t)\qquad(0\leqq t\leqq\theta),$$

$$\int_{c_1}\alpha:=\int_0^\theta\left(a(x_1(t),y_1(t))\,\frac{dx_1}{dt}\,dt+b(x_1(t),y_1(t))\,\frac{dy_1}{dt}\,dt\right)$$

$=$

θ



Exercise 1-2; Line Integral (2)

$$U = \mathbb{R}^2 \setminus \{(t,0)\,;\, t \leqq 0\},\ \mathrm{P} = (r\cos\theta, r\sin\theta)$$

defined on $\hat{U} = \{(x,y) \mid (x,y) \neq (0,0)\}$

$$\underline{\alpha = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy}$$

$$c_2(s) := \big(x_2(s), y_2(s)\big) = (s\cos\theta, s\sin\theta) \qquad (1 \leqq s \leqq r),$$

$$\int_{c_2} \alpha := \int_1^r \left(a(x_2(s), y_2(s))\,\frac{dx_2}{ds}\,ds + b(x_2(s), y_2(s))\,\frac{dy_2}{ds}\,ds \right).$$

$$= 0$$

$\nexists f$ s.t $df = \alpha$ on $\hat{U}$

- We assumed that $r>1$, $\theta\in(0,\pi)$
- It works for $0<r\le 1$ & $-\pi<\theta\le 0$

$$P=(x,y)=(r\cos\theta, r\sin\theta)$$

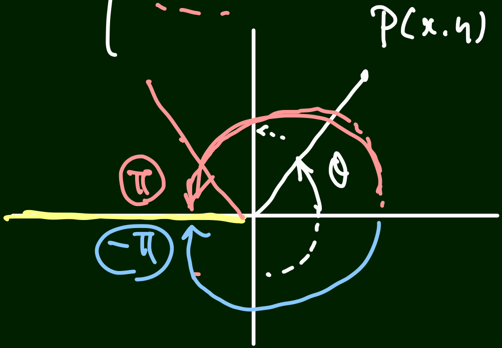
$(x,y)\in U$

$$\int_{c_1\cup c_2}\alpha=\theta=\theta(x,y)=\begin{cases}\tan^{-1}\frac{y}{x} & (\text{if } -\frac{\pi}{2}<\theta<\frac{\pi}{2})\\ \pi-\tan\frac{x}{y} & (\text{if } \frac{\pi}{2}\le\theta<\pi)\\ \cdots & \end{cases}$$

$$\Rightarrow \boxed{df=\alpha}$$

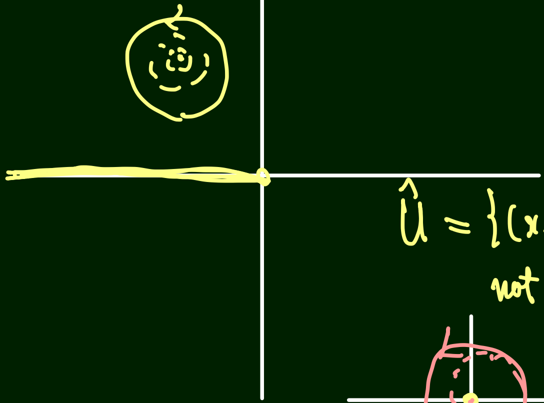
$\tan^{-1}u\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

not continuous on $\hat{U}$



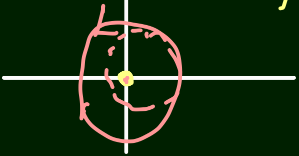
Simple connectedness of the region: all loops are deformed to a point

U



$\hat{U} = \{(x,y) \mid (x,y) \neq (0,0)\}$

not simply connected.



Ordinary Differential Equations 常微分方程式

(normal form of) known function in t & x

$$\boxed{\frac{d}{dt}x(t) = f(t, x(t)),} \quad \boxed{x(t_0) = x_0} \quad \text{initial condition } (*)$$

extra parameter

$x(t)$: unknown function
$\mathbb{R}^n$-valued

- Existence
- Uniqueness
- Regularity on initial conditions and parameters

sol. is smooth in extra param.

$\exists^1 x : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow \mathbb{R}^n$ satisfying $(*)$

Fundamental Theorem of O.D.E.

Example

$t \in \mathbb{R}$

$f(t, x) = \lambda x$

x: $\mathbb{R}$-valued

λ: const.

- $\frac{d}{dt}x(t) = f(t, x(t)) = \lambda x(t)$, $\quad x(0) = x_0$.

initial condition

$$\boxed{\frac{dx}{dt} = \lambda x}$$

$$\boxed{x(t) = \text{const} \cdot e^{\lambda t}}$$

general solution

uniqueness of solution?

$x(t) = x_0 e^{\lambda t}$

the unique sol.

Example

$t \in \mathbb{R}$

ω: const

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \cos \omega t + \frac{y_0}{\omega} \sin \omega t \\ -x_0 \omega \sin \omega t + y_0 \cos \omega t \end{pmatrix}$$

$$\boxed{\frac{d}{dt} \mathbf{x} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \mathbf{x}}$$

$\mathbf{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\omega^2 x \end{cases}$$

$$\Rightarrow \left(\frac{d^2 x}{dt^2} = -\omega^2 x \right)$$

Example

$f(t,x) = t(1+x^2)$

smaller interval

$\boxed{-\sqrt{\pi} < t < \sqrt{\pi}}$

$$\frac{dx}{dt} = t(1+x^2), \qquad x(0) = 0.$$

$\boxed{x = \tan \frac{1}{2} t^2}$

$\frac{d}{du} \tan u = 1 + \tan^2 u$

$$\frac{dx}{dt} = \left(1 + \tan^2 \frac{1}{2} t^2\right) \cdot t$$

$$= (1+x^2) t$$

By uniqueness, this is the only solution

Difference with previous 2-equations: Linearlity.