Advanced Topics in Geometry A1 (MTH.B405)

Ordinary Differential Equations

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Linear ordinary differential equations

linear function in unknowns

$$(*)\ \begin{cases} \dfrac{d}{dt}\boldsymbol{x}(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t), \\ x(t_0) = x_0 \end{cases}$$

$\mathbb{R}^n$-valued

- Global Existence

$x(t)$: $\mathbb{R}^n$-valued

$A(t)$: $n \times n$ matrix valued functions

$A(t)$, $\mathbb{b}(t)$: given C^∞-functions on I (possibly $\mathbb{R}$)

$\Rightarrow \exists\, x : I \longrightarrow \mathbb{R}^n$ satisfying $(*)$

Proof : not hard

For consistency, it is introduced in Lecture note.

Linear ordinary differential equations in matrix forms

matrix-valued unknowns.

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0,$$

$\Omega(t)$, $B(t)$: $n \times n$-matrix valued C^∞-functions.

(given data)

$X(t)$: unknown function $n \times n$ matrix-valued.

X_0: the initial condition

Preliminaries

homogenious equation

Proposition (Prop. 2.8)

Assume two C^∞ matrix-valued functions $X(t)$ and $\Omega(t)$ satisfy

B = 0

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0.$$

Then

$$\det X(t) = (\det X_0) \exp \int_{t_0}^{t} \operatorname{tr} \Omega(\tau)\, d\tau.$$

In particular, if $X_0 \in \mathrm{GL}(n, \mathbb{R})$, then $X(t) \in \mathrm{GL}(n, \mathbb{R})$ for all t.

- $\frac{d}{dt} X = X\Omega \qquad X(0) = X_0$

- $\frac{d}{dt}(\det X) = \operatorname{trace}\left(\tilde{X} \frac{dX}{dt}\right)$

$= \operatorname{trace} \tilde{X} X \Omega$

$= (\det X) \cdot \operatorname{trace} \Omega$

余因子

- $\tilde{X}$: the cofactor matrix
- $\tilde{X} X = X \tilde{X}$
  $= (\det X)\, id$
- if X : non-singular
  $\Rightarrow \tilde{X} = (\det X) \cdot X^{-1}$

Preliminaries

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0.$$

Corollary (Cor. 2.9)

If $\operatorname{tr}\Omega(t) = 0$, *then* $\det X(t)$ *is constant. In particular, if* $X_0 \in \mathrm{SL}(n, \mathbb{R})$, X *is a function valued in* $\mathrm{SL}(n, \mathbb{R})$.

(n×n-matrices with real components
with det = 1)

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0.$$

$XX^T = \mathrm{id}$

Proposition (Prop. 2.10)

Assume $\Omega^T + \Omega = O$.
If $X_0 \in \mathrm{O}(n)$ (*resp.* $X_0 \in \mathrm{SO}(n)$),
then $X(t) \in \mathrm{O}(n)$ (*resp.* $X(t) \in \mathrm{SO}(n)$) *for all* t.

Ω is skew-symmetric.

O(n) : orthogonal matrices

$$\frac{d}{dt}(XX^T) = \frac{dX}{dt}X^T + X\left(\frac{dX}{dt}\right)^T$$
$$= X\Omega X^T + X(X\Omega)^T$$
$$= X(\Omega + {}^T\Omega)X^T = O$$

$X(t) \in O(n)$

Linear ordinary differential equations.

Proposition (Prop. 2.12)

Let $\Omega(t)$ *be a* C^∞*-function valued in* $\mathrm{M}_n(\mathbb{R})$ *defined on an interval* I*. Then for each* $t_0 \in I$*, there exists the unique matrix-valued* C^∞*-function* $X(t) = X_{t_0,\mathrm{id}}(t)$ *such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = \mathrm{id}\,.$$

Linear ordinary differential equations.

Corollary (Cor. 2.13)

There exists the unique matrix-valued C^∞-function $X_{t_0,X_0}(t)$ defined on I such that

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \quad X(t_0) = X_0 \quad \big(X(t) := X_{t_0,X_0}(t)\big)$$

In particular, $X_{t_0,X_0}(t)$ is of class C^∞ in X_0 and t.

Non-homogenious case

Proposition (Prop. 2.14)

Let $\Omega(t)$ *and* $B(t)$ *be matrix-valued* C^∞*-functions defined on* I*. Then for each* $t_0 \in I$ *and* $X_0 \in \mathrm{M}_n(\mathbb{R})$*, there exists the unique matrix-valued* C^∞*-function defined on* I *satisfying*

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0.$$

Fundamental Theorem

Theorem (Thm. 2.15)

Let I and U be an interval and a domain in $\mathbb{R}^m$, respectively, and let $\Omega(t, \boldsymbol{\alpha})$ and $B(t, \boldsymbol{\alpha})$ be matrix-valued C^∞-functions defined on $I \times U$ ($\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$). Then for each $t_0 \in I$, $\boldsymbol{\alpha} \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0, X_0, \boldsymbol{\alpha}}(t)$ defined on I such that

$$\frac{dX(t)}{dt} = X(t)\Omega(t, \boldsymbol{\alpha}) + B(t, \boldsymbol{\alpha}), \qquad X(t_0) = X_0. \tag{1}$$

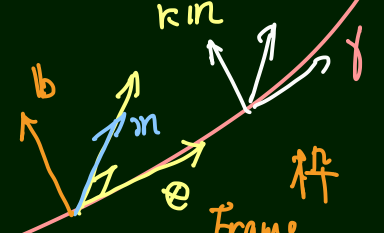
"solution is smooth in α"

Moreover,

$$I \times I \times \mathrm{M}_n(\mathbb{R}) \times U \ni (t, t_0, X_0, \boldsymbol{\alpha}) \mapsto X_{t_0, X_0, \boldsymbol{\alpha}}(t) \in \mathrm{M}_n(\mathbb{R})$$

is a C^∞-map.

- $\gamma\colon I \to \mathbb{R}^3$: a space curve parametrized by the arclength.
- $\boldsymbol{e} = \gamma'$.
- $\kappa = |\boldsymbol{e}'|$; we assume $\kappa > 0$ (the curvature)
- $\boldsymbol{n} = \boldsymbol{e}'/\kappa$ (the principal normal)
- $\boldsymbol{b} = \boldsymbol{e} \times \boldsymbol{n}$ (the binormal)
- $\tau = -\boldsymbol{b}' \cdot \boldsymbol{n}$ (the torsion)

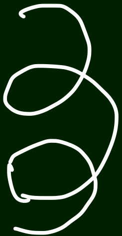


Example

$$\gamma(t) = \left(a\cos t \quad a\sin t \quad bt\right)$$

helix

$$\hat{\gamma}(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}} \quad a\sin\frac{s}{\sqrt{a^2+b^2}} \quad \frac{b}{\sqrt{a^2+b^2}}s\right)$$



helix

κ, τ: constant

Frenet-Serret

- $\mathcal{F} := (\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{b})\colon I \to \mathrm{SO}(3)$: the Frenet Frame

$$\frac{d\mathcal{F}}{ds} = \mathcal{F}\Omega, \qquad \Omega = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

skew symm.

$\gamma'(s) = e(s)$

Procedure to recover the curve from κ & τ.

κ, τ: given function

$\Rightarrow$ Linear O.D.E

by Fundamental theorems.

$\Rightarrow \exists$ solution

$\Rightarrow \mathcal{F}\colon I \to SO(3)$

$= (e, n, b)$

The Fundamental Theorem for Space Curves

Theorem (Thm. 2.17)

functions

Let $\kappa(s)$ and $\tau(s)$ be C^∞-fnctions defined on an interval I satisfying $\kappa(s) > 0$ on I.
Then there exists a space curve $\gamma(s)$ parametrized by arc-length whose curvature and torsion are κ and τ, respectively.
Moreover, such a curve is unique up to transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) of $\mathbb{R}^3$.

- κ & τ determine a space curve.

Exercise 2-1

Problem (Ex. 2-1)

Logistic equation

Find the maximal solution of the initial value problem

$$\frac{dx}{dt} = x(1 - x), \qquad x(0) = a.$$

where a is a real number.

$0 < a < 1$

$a = 0$

$a = 1$ $\quad a < 0$ $\quad a > 1$

Exercise 2-2

Problem (Ex. 2-2)

pendulum

Let $x = x(t)$ *be the maximal solution of an initial value problem of differential equation*

$$\frac{d^2x}{dt^2} = -\sin x, \qquad x(0) = 0, \quad \frac{dx}{dt}(0) = 2.$$

- *Show that* $\dfrac{dx}{dt} = 2\cos\dfrac{x}{2}$.
- *Verify that* x *is defined on* $\mathbb{R}$*, and compute* $\lim_{t\to\pm\infty} x(t)$.

Exercise 2-3

Problem (Ex. 2-3)

Find an explicit expression of a space curve $\gamma(s)$ parametrized by the arc-length s, whose curvature κ and torsion τ satisfy

$$\kappa = \tau = \frac{1}{\sqrt{2}(1+s^2)}.$$

$$\frac{d\mathcal{F}}{ds} = \frac{1}{\sqrt{2}(1+s^2)}\mathcal{F}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

changing parameter

$$\frac{d\mathcal{F}}{dt} = \mathcal{F}\begin{pmatrix} \quad \end{pmatrix}$$