Advanced Topics in Geometry A1 (MTH.B405)

Ordinary Differential Equations

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Linear ordinary differential equations

$$\frac{d}{dt}\boldsymbol{x}(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t),$$

- Global Existence

Linear ordinary differential equations in matrix forms

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0,$$

Preliminaries

Proposition (Prop. 2.8)

Assume two C^∞ *matrix-valued functions* $X(t)$ *and* $\Omega(t)$ *satisfy*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0.$$

Then

$$\det X(t) = (\det X_0) \exp \int_{t_0}^{t} \operatorname{tr} \Omega(\tau)\, d\tau.$$

In particular, if $X_0 \in \mathrm{GL}(n, \mathbb{R})$*,then* $X(t) \in \mathrm{GL}(n, \mathbb{R})$ *for all* t*.*

Preliminaries

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0.$$

Corollary (Cor. 2.9)

If $\operatorname{tr}\Omega(t) = 0$*, then* $\det X(t)$ *is constant. In particular, if* $X_0 \in \mathrm{SL}(n, \mathbb{R})$*,* X *is a function valued in* $\mathrm{SL}(n, \mathbb{R})$*.*

Preliminaries

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0.$$

Proposition (Prop. 2.10)

Assume $\Omega^T + \Omega = O$.
If $X_0 \in \mathrm{O}(n)$ (*resp.* $X_0 \in \mathrm{SO}(n)$),
then $X(t) \in \mathrm{O}(n)$ (*resp.* $X(t) \in \mathrm{SO}(n)$) *for all* t.

Linear ordinary differential equations.

Proposition (Prop. 2.12)

Let $\Omega(t)$ be a C^∞-function valued in $\mathrm{M}_n(\mathbb{R})$ defined on an interval I. Then for each $t_0 \in I$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,\mathrm{id}}(t)$ such that

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = \mathrm{id}\,.$$

Linear ordinary differential equations.

Corollary (Cor. 2.13)

There exists the unique matrix-valued C^∞-function $X_{t_0,X_0}(t)$ defined on I such that

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \quad X(t_0) = X_0 \quad \big(X(t) := X_{t_0,X_0}(t)\big)$$

In particular, $X_{t_0,X_0}(t)$ is of class C^∞ in X_0 and t.

Non-homogenious case

Proposition (Prop. 2.14)

Let $\Omega(t)$ and $B(t)$ be matrix-valued C^∞-functions defined on I. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function defined on I satisfying

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0.$$

Fundamental Theorem

Theorem (Thm. 2.15)

Let I and U be an interval and a domain in $\mathbb{R}^m$, respectively, and let $\Omega(t, \boldsymbol{\alpha})$ and $B(t, \boldsymbol{\alpha})$ be matrix-valued C^∞-functions defined on $I \times U$ ($\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$). Then for each $t_0 \in I$, $\boldsymbol{\alpha} \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0, X_0, \boldsymbol{\alpha}}(t)$ defined on I such that

$$\frac{dX(t)}{dt} = X(t)\Omega(t, \boldsymbol{\alpha}) + B(t, \boldsymbol{\alpha}), \qquad X(t_0) = X_0. \tag{1}$$

Moreover,

$$I \times I \times \mathrm{M}_n(\mathbb{R}) \times U \ni (t, t_0, X_0, \boldsymbol{\alpha}) \mapsto X_{t_0, X_0, \boldsymbol{\alpha}}(t) \in \mathrm{M}_n(\mathbb{R})$$

is a C^∞-map.

Application to Space Curves

- $\gamma\colon I \to \mathbb{R}^3$: a space curve parametrized by the arclength.
- $\boldsymbol{e} = \gamma'$
- $\kappa = |\boldsymbol{e}'|$; we assume $\kappa > 0$ (the curvature)
- $\boldsymbol{n} = \boldsymbol{e}'/\kappa$ (the principal normal)
- $\boldsymbol{b} = \boldsymbol{e} \times \boldsymbol{n}$ (the binormal)
- $\tau = -\boldsymbol{b}' \cdot \boldsymbol{n}$ (the torsion)

Frenet-Serret

- $\mathcal{F} := (\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{b})\colon I \to \mathrm{SO}(3)$: the Frenet Frame

$$\frac{d\mathcal{F}}{ds} = \mathcal{F}\Omega, \qquad \Omega = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

The Fundamental Theorem for Space Curves

Theorem (Thm. 2.17)

Let $\kappa(s)$ and $\tau(s)$ be C^∞-fnctions defined on an interval I satisfying $\kappa(s) > 0$ on I.
Then there exists a space curve $\gamma(s)$ parametrized by arc-length whose curvature and torsion are κ and τ, respectively.
Moreover, such a curve is unique up to transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ $(A \in \mathrm{SO}(3), \boldsymbol{b} \in \mathbb{R}^3)$ of $\mathbb{R}^3$.

Exercise 2-1

Problem (Ex. 2-1)

Find the maximal solution of the initial value problem

$$\frac{dx}{dt} = x(1-x), \qquad x(0) = a,$$

where a is a real number.

Exercise 2-2

Problem (Ex. 2-2)

Let $x = x(t)$ *be the maximal solution of an initial value problem of differential equation*

$$\frac{d^2x}{dt^2} = -\sin x, \qquad x(0) = 0, \quad \frac{dx}{dt}(0) = 2.$$

- *Show that* $\frac{dx}{dt} = 2\cos\frac{x}{2}$.
- *Verify that* x *is defined on* $\mathbb{R}$, *and compute* $\lim_{t\to\pm\infty} x(t)$.

Exercise 2-3

Problem (Ex. 2-3)

Find an explicit expression of a space curve $\gamma(s)$ *parametrized by the arc-length* s*, whose curvature* κ *and torsion* τ *satisfy*

$$\kappa = \tau = \frac{1}{\sqrt{2}(1+s^2)}.$$