Advanced Topics in Geometry A1 (MTH.B405)

Integrability Conditions

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Problem 2-1

Logistic equation

(limit case of SIR-model)

Problem (Ex. 2-1)

Find the maximal solution of the initial value problem

$$\frac{dx}{dt} = x(1-x), \qquad x(0) = a,$$

where a is a real number.

- When $a = 0, 1,$ $\begin{cases} x(t) = 0 \\ x(t) = 1 \end{cases}$: are solutions
-

When $a \in \mathbb{R} \setminus \{0, 1\}$

$$\frac{1}{\underbrace{x(1-x)}_{(\neq 0 \text{ near } t=0)}} \frac{dx}{dt} = 1 \qquad x(0) = a$$

$$\int_0^t dt$$

$$\int_0^t \frac{1}{x(1-x)} \frac{dx}{dt} \, dt = t$$

$$= \int_{x(0)}^{x(t)} \frac{1}{x(1-x)} \, dx = \Big[\log \left| \frac{x}{1-x} \right| \Big]_a^{x(t)}$$

$$= \log \left(\frac{x}{1-x} \cdot \frac{1-a}{a} \right)$$

same sign near $t = 0$

The logistic equation

$x' = x(1-x), \qquad x(0) = a$ The Solution: $\boxed{x(t) = \dfrac{1}{1 + \underbrace{\frac{1-a}{a}}_{0 <} e^{-t}}}$

When $a \in (0, 1)$

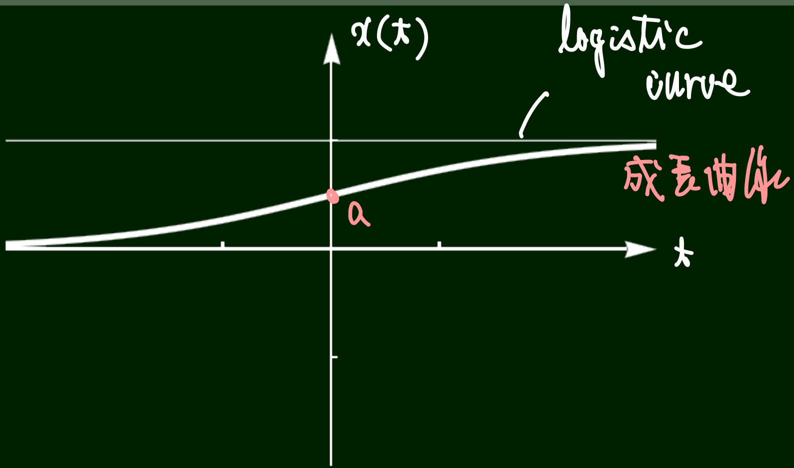
x is defined on $\mathbb{R}$,

When $a > 1$

$x(t)$ -- : $\log \frac{a-1}{a} < t < +\infty$

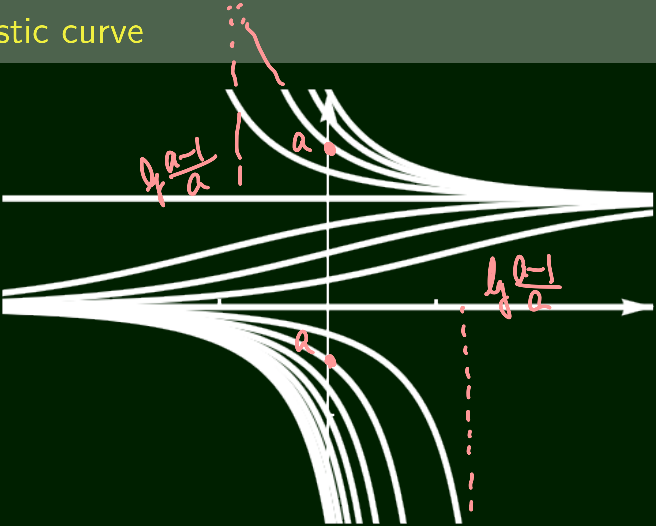
$a < 0$: $-\infty < t < \log \frac{a-1}{a}$

The logistic curve



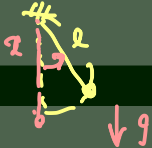
$$x(t) = \frac{1}{1 + \frac{1-a}{a} e^{-t}} \ (a \in (0,1))$$

The logistic curve



Exercise 2-2

pendulum



Problem (Ex. 2-2)

Let $x = x(t)$ be the maximal solution of an initial value problem of differential equation

$$\frac{d^2x}{dt^2} = -\sin x, \qquad \underline{x(0) = 0, \quad \frac{dx}{dt}(0) = 2.}$$

- *Show that* $\dfrac{dx}{dt} = 2\cos\dfrac{x}{2}$.
- *Verify that x is defined on $\mathbb{R}$, and compute* $\lim_{t\to\pm\infty} x(t)$.

<u>Note</u> The only solution of eq. of pendulum in "elementary functions"

$$\frac{dx}{dt}\frac{d^2x}{dt^2} = -\sin x \frac{dx}{dt} \quad \rightsquigarrow \quad \frac{1}{2}\frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = \frac{d}{dt}\cos x$$

$$\frac{d}{dt}\left(\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \cos x\right) = 0$$

The equation of motion of pendulums

energy integral

· conservation law of mechanical energy

$$\frac{d^2x}{dt^2} + \sin x = 0 \qquad \Rightarrow \qquad \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \cos x = \text{const.}$$

$x(0) = 0 \qquad \dot{x}(0) = 2$

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \cos x = \text{const.}$$

$$= \frac{1}{2}\left(\underbrace{\frac{dx}{dt}(0)}_{2}\right)^2 - \cos \underbrace{x(0)}_{0} = \frac{1}{2} \times 2^2 - 1 = 1$$

$$(\because \quad x(0) = 0, \quad \dot{x}(0) = 2.)$$

$\Rightarrow$

$$\left(\frac{dx}{dt}\right)^2 = 2(1 + \cos x) = 4\cos^2\frac{x}{2}.$$

$$\boxed{\frac{dx}{dt} = 2\cos\frac{x}{2}}$$

$+$

$\frac{d}{dt}x(0) = 2 > 0$

$x(0) = 1$

$$\frac{dx}{dt} = 2\cos\frac{x}{2} \qquad \Rightarrow \qquad 1 = \frac{1}{2}\sec\frac{x}{2}\frac{dx}{dt}.$$

$$t = \int_0^t 1\,dt = \frac{1}{2}\int_0^t \sec\frac{x(t)}{2}\frac{dx(t)}{dt}\,dt = \frac{1}{2}\int_{x(0)}^{x(t)} \sec\frac{x}{2}\,dx$$

$$= \log\frac{1+\tan\frac{x}{4}}{1-\tan\frac{x}{4}}$$

$$\tan\frac{x}{4} = \frac{e^t-1}{e^t+1}$$

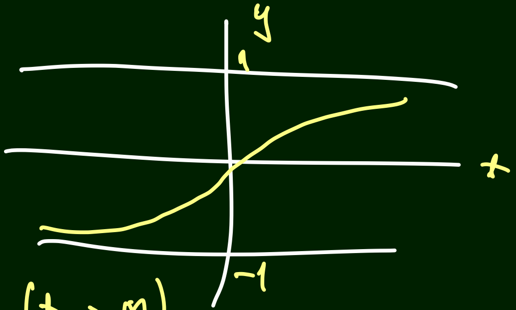
$$= \tanh\frac{t}{2}$$

$$\boxed{x(t) = 4\tan^{-1}\tanh\frac{t}{2}}$$

$$\boxed{x(t) = 4\tan^{-1}\tanh\frac{t}{2}}$$

pseudosphere

$y = \tanh\frac{t}{2}$



$x \to \pi \quad (t \to \infty)$

$x \to -\pi \quad (t \to -\infty)$

Exercise 2-3

linear differential eq.

Problem (Ex. 2-3)

Find an explicit expression of a space curve $\gamma(s)$ parametrized by the arc-length s, whose curvature κ and torsion τ satisfy

$$\kappa = \tau = \frac{1}{\sqrt{2}(1+s^2)}.$$

Strategy: Solve

$$\frac{d}{ds}\mathcal{F} = \mathcal{F}\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}, \qquad \mathcal{F}(0) = \mathrm{id}$$

for $\mathcal{F}(s) = (\boldsymbol{e}(s), \boldsymbol{n}(s), \boldsymbol{b}(s))$.

Then the desired curve is obtained by

$\gamma'(s)$

$$\gamma(s) = \int_0^s \boldsymbol{e}(s)\,ds \quad \rightarrow$$

$\kappa = \tau = \frac{1}{\sqrt{2}} \frac{1}{1+s^2}$

$$\frac{d}{ds}\mathcal{F} = \mathcal{F}\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} = \frac{1}{1+s^2}\mathcal{F}\Omega, \qquad \Omega := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

constant matrix

3×3 det $= 0$, tr $= 0$

Change Variable $\boxed{s = \tan t} \Rightarrow$

$$\boxed{\frac{d}{dt}\mathcal{F} = \mathcal{F}\Omega}$$

const

$\frac{ds}{dt} = 1 + s^2$

With initial condition $\mathcal{F}(0) = \mathrm{id}$,

$$\mathcal{F}(t) = \exp t\Omega = \mathrm{id} + \sum_{k=1}^{\infty} \frac{t^k}{k!}\Omega^k$$

Cayley-Hamilton

$\Omega^3 + \cancel{*}\Omega = 0$

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Frenet-Serret equation

$$\mathcal{F}(t) = \exp t\Omega = \mathrm{id} + \sum_{k=1}^{\infty} \frac{t^k}{k!}\Omega^k \qquad \Omega := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Note:

$$\Omega^2 = \frac{1}{2}\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \qquad \Omega^3 = -\Omega, \qquad \Omega^4 = -\Omega^2.$$

$\Rightarrow$

$s = \tan t$

$$\exp t\Omega = \begin{pmatrix} \frac{1}{2}(1+\cos t) & * & * \\ \frac{1}{\sqrt{2}}\sin t & * & * \\ \frac{1}{2}(1-\cos t) & * & * \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(1 + \frac{1}{\sqrt{1+s^2}}\right) & * & * \\ \frac{1}{\sqrt{2}}\frac{s}{\sqrt{1+s^2}} & * & * \\ \frac{1}{\sqrt{2}}\left(1 - \frac{1}{\sqrt{1+s^2}}\right) & * & * \end{pmatrix}$$

Frenet-Serret equation

$$
\boldsymbol{e}(s) = \gamma'(s) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(1 + \frac{1}{\sqrt{1+s^2}}\right) \\ \frac{1}{\sqrt{2}}\frac{s}{\sqrt{1+s^2}} \\ \frac{1}{\sqrt{2}}\left(1 - \frac{1}{\sqrt{1+s^2}}\right) \end{pmatrix}
$$

$$
\gamma(s) = \begin{pmatrix} \frac{1}{2}\left(s + \log(s + \sqrt{1+s^2})\right) \\ \frac{1}{\sqrt{2}}\sqrt{1+s^2} \\ \frac{1}{2}\left(s - \log(s + \sqrt{1+s^2})\right) \end{pmatrix}
$$

Frenet-Serret equation: an alternative solution

$$\frac{d}{dt}\left(\frac{e+b}{\sqrt{2}}\right) = 0$$

const

✓ $$\frac{d}{ds}\mathcal{F} = \mathcal{F}\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\kappa \\ 0 & \kappa & 0 \end{pmatrix} \quad \because \quad \kappa = \tau$$

$$\frac{-e+b}{\sqrt{2}}$$

Set

$$\mathcal{G} = \mathcal{F}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

then

$$\boxed{\frac{d}{ds}\mathcal{G} = \mathcal{G}\Lambda,} \qquad \Lambda := \frac{1}{1+s^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$