Advanced Topics in Geometry A1 (MTH.B405)

Integrability Conditions

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Integrability Conditions

$X = X(u_1 \cdots u_m)$

linear

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \ldots, m), \qquad X(\mathrm{P}_0) = X_0. \qquad (*)$$

$n \times n$ real matrices, $\det \neq 0$

Proposition (Prop. 3.2)

If a matrix-valued C^∞ function $X : U \to \mathrm{GL}(n, \mathbb{R})$ satisfies $(*)$, it holds that

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j \Omega_k - \Omega_k \Omega_j$$

for each (j, k) with $1 \leqq j < k \leqq m$.

$X = X(u,v)$: $n\times n$-matrix valued

$$\frac{\partial X}{\partial u} = X\Omega \qquad \frac{\partial X}{\partial v} = X\Lambda$$

Given

Prop Thm 3.2 Ω & Λ : functions in (u,v)

If X is a solution, valued in $GL(n,\mathbb{R})$

$$\Rightarrow \quad \frac{\partial \Omega}{\partial v} - \frac{\partial \Lambda}{\partial u} = \Omega\Lambda - \Lambda\Omega$$

☺ $$\frac{\partial^2 X}{\partial v \partial u} = \frac{\partial X}{\partial v}\Omega + X\frac{\partial \Omega}{\partial v} = X\Lambda\Omega + X\frac{\partial \Omega}{\partial v}$$

$$\frac{\partial^2 X}{\partial v \partial u} = \frac{\partial X}{\partial v}\Omega + X\frac{\partial \Omega}{\partial v} = X\Lambda\Omega + X\frac{\partial \Omega}{\partial v}$$

$$\parallel$$

$$\frac{\partial^2 X}{\partial u \partial v} = \frac{\partial X}{\partial u}\Lambda + X\frac{\partial \Lambda}{\partial u} = X\Omega\Lambda + X\frac{\partial \Lambda}{\partial u}$$

$$X\left(\Lambda\Omega + \frac{\partial \Omega}{\partial v}\right) = X\left(\Omega\Lambda + \frac{\partial \Lambda}{\partial u}\right)$$

$$(\det X \neq 0)$$

Integrability of Linear systems

- $$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0. \tag{1}$$
- $$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j\Omega_k - \Omega_k\Omega_j \tag{2}$$

Disc, whole space : simply connected.

Theorem (Thm. 3.5)

Let $\Omega_j : U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ be C^∞-functions defined on a simply connected domain $U \subset \mathbb{R}^m$ satisfying (2). Then for each $\mathrm{P}_0 \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique $n \times n$-matrix valued function $X : U \to \mathrm{M}_n(\mathbb{R})$ satisfying (1)

$U \subset \mathbb{R}^2$ simply connected domain (homeo. to disc)

· Ω, Λ: matrix valued functions defined on U C^∞

with $\Omega_v - \Lambda_u = \Omega\Lambda - \Lambda\Omega$ integrability conditions

· $P_0 \in U$, X_0: a given matrix

$\Rightarrow$ $\exists ! \, X : U \longrightarrow M_n(\mathbb{R})$: $n \times n$ matrices

with $\left(\dfrac{\partial X}{\partial u} = X\Omega \quad \dfrac{\partial X}{\partial v} = X\Lambda \quad X(P_0) = X_0 \right)$

Integrability Conditions

Lemma (Lem. 3.4)

Let $\Omega_j \colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \ldots, m)$ *be* C^∞*-maps defined on a domain* $U \subset \mathbb{R}^m$ *which satisfy*

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j \Omega_k - \Omega_k \Omega_j.$$

Then for each smooth map

$$\sigma \colon D \ni (t, w) \longmapsto \sigma(t, w) = (u^1(t, w), \ldots, u^m(t, w)) \in U$$

defined on a domain $D \subset \mathbb{R}^2$*, it holds that*

$$\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} - TW + WT = 0,$$

where $T := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial t}$, $W := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial w}$, $(\widetilde{\Omega}_j := \Omega_j \circ \sigma)$.

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0. \qquad (*)$$

Lemma (Lem. 3.3)

Let $X\colon U \to \mathrm{M}_n(\mathbb{R})$ *be a* C^∞*-map satisfying* $(*)$ *Then for each smooth path* $\gamma\colon I \to U$ *defined on an interval* $I \subset \mathbb{R}$*,* $\hat{X} := X \circ \gamma : I \to \mathrm{M}_n(\mathbb{R})$ *satisfies the ordinary differential equation*

$$\frac{d\hat{X}}{dt}(t) = \hat{X}(t)\Omega_\gamma(t) \quad \left(\Omega_\gamma(t) := \sum_{j=1}^{n} \Omega_j \circ \gamma(t)\frac{du^j}{dt}(t)\right)$$

on I*, where* $\gamma(t) = \bigl(u^1(t), \dots, u^m(t)\bigr)$*.*

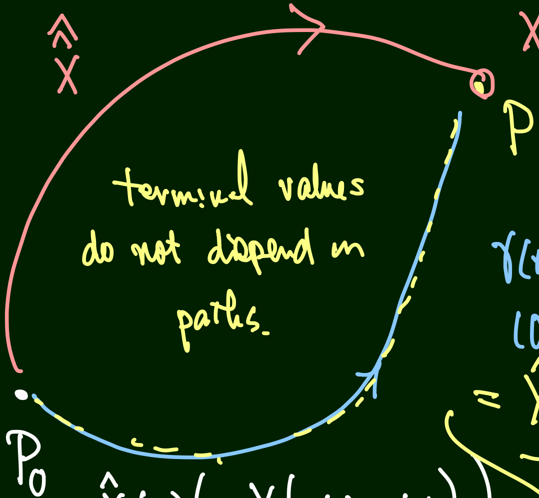
Proof of Theorem 2.5 (outline)

- Take $\mathrm{P} \in U$, and a path $\gamma\colon [0,1] \to U$ with $\gamma(0) = \mathrm{P}_0$ and $\gamma(1) = \mathrm{P}$.
- Solve the linear ODE as in Lemma 2.3 with initial condition $\hat{X}(0) = X_0$.
- Show the value $\hat{X}(1)$ does not depend on $\gamma \Leftarrow$ by Lem. 3.4
- Define $X(\mathrm{P}) := \hat{X}(1)$.
- Show X is the desired solution.

U

- $X : U \to M_n(\mathbb{R})$

$X(P) = \hat{X}(1) \overset{?}{=} \hat{\hat{X}}(1)$



$(X(P_0) = X_0)$

$\gamma(t) = (u(t), v(t))$

$(0 \leqq t \leqq 1)$

$\hat{X}(t) \,(= X(u(t), v(t)))$

$$= \hat{X}\left(\Omega \frac{du}{dt} + \Lambda \frac{dv}{dt}\right)$$

$$\hat{X}(0) = P_0$$

ODE

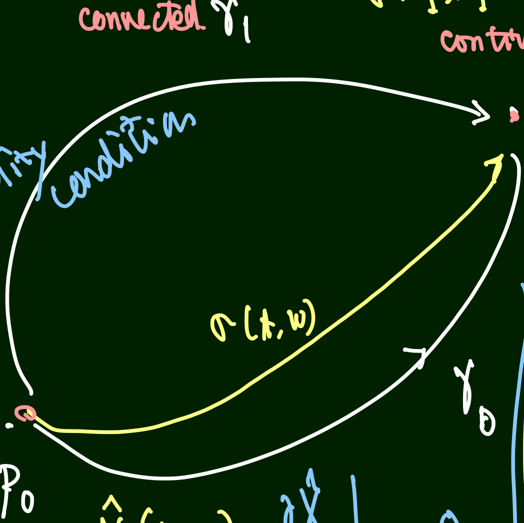
(*) $$\frac{d\hat{X}}{dt} = \frac{\partial X}{\partial u}\frac{du}{dt} + \frac{\partial X}{\partial v}\frac{dv}{dt}$$

$I = [0,1]$ U: simply connected $\Longrightarrow$ $\exists\ \sigma : I \times I \to U$ (C^∞)

U)

continuously deform from γ_0 to γ_1

Whitney.



Integrability condition

$$\hat{X}(t,w) \quad \left.\frac{\partial \hat{X}}{\partial w}\right|_{t=0,\ t=1} = 0$$

$$\sigma(t,0) = \gamma_0(t)$$
$$\sigma(t,1) = \gamma_1(t)$$
$$\sigma(0,w) = P_0$$
$$\sigma(1,w) = P$$

Application: Poincaré's lemma

Theorem (Poincaré's lemma)

If a differential 1*-form*

$$\omega = \sum_{j=1}^{m} \alpha_j(u^1, \dots, u^m)\, du^j$$

defined on a simply connected domain $U \subset \mathbb{R}^m$ *is closed, that is,* $d\omega = 0$ *holds, then there exists a* C^∞*-function* f *on* U *such that* $df = \omega$*. Such a function* f *is unique up to additive constants.*

$\alpha = a(u,v)\,du + b(u,v)\,dv$ $\qquad U$: simpl. conn.

$d\alpha = (b_u - a_v)\,du \wedge dv = 0$

$\Rightarrow \exists f(u,v)$ s.t. $df = f_u\,du + f_v\,dv = \alpha$

- $\left(\sigma_u = a\sigma \quad \sigma_v = b\sigma \quad \sigma(P_0) = 1 \right)$ $\qquad \sigma$: 1×1-matrix valued

$\sigma_{uv} = \sigma a_v + \sigma_v a = \sigma a_v + \sigma ab$, $\quad \sigma_{vu} = \sigma b_u + \sigma_u b = \sigma b_u + \sigma ba$

integrability $\quad a_v - b_u = \cancel{ab - ba} = 0$

$\exists\, \sigma : U \longrightarrow \mathbb{R} \qquad \cdot\ \sigma > 0$

$\boxed{f := \log \sigma}$

Application: Conjugation of harmonic functions

Theorem

Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u,v)$ a C^∞-function harmonic on U. Then there exists a C^∞ harmonic function η on U such that $\xi(u,v) + \mathrm{i}\,\eta(u,v)$ is holomorphic on U.

$\Delta\xi = \xi_{uu} + \xi_{vv} = 0$

η : conjugate of ξ

$\Rightarrow$

$$\begin{cases} \eta_u = -\xi_v \\ \eta_v = \xi_u \end{cases}$$

Cauchy–Riemann

integrability condition:

$-\xi_{vv} = \xi_{uu}$ i.e. $\Delta\xi = 0$

Application: Conjugation of harmonic functions

Example

$\xi(u, v) = e^u \cos v$

$\eta(u, v) = e^u \sin v$

$$\xi + i\eta = e^u(\cos v + i \sin v) = e^{u+iv}$$

Exercise 3-1

Problem

Let

$$\xi_1(u,v) := \frac{u}{u^2+v^2}, \qquad \xi_2(u,v) := \log\sqrt{u^2+v^2}$$

be a function defined on non-simply connected domain $U := \mathbb{R}^2 \setminus \{(0,0)\}$.

1. *Show that both ξ_1 and ξ_2 are harmonic on U.*
2. *Verify that there exists a conjugate harmonic functaion η_1 of ξ_1 on U.*
3. *Prove that there exists no conjugate harmonic functaion η_2 of ξ_2 on U.*

Exercise 3-2

$\Omega_v - \Lambda_u = \Omega\Lambda - \Lambda\Omega$

Problem

Consider a linear system of partial differential equations for 3×3*-matrix valued unknown* X *on a domain* $U \subset \mathbb{R}^2$ *as*

$$\frac{\partial X}{\partial u} = X\Omega, \quad \frac{\partial X}{\partial v} = X\Lambda,$$

$$\left(\Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix}\right),$$

where (u, v) *are the canonical coordinate system of* $\mathbb{R}^2$*, and* α*,* β *and* h_j^i *(*$i, j = 1, 2$*) are smooth functions defined on* U*. Write down the integrability conditions in terms of* α*,* β *and* h_j^i*.*