Advanced Topics in Geometry A1 (MTH.B405)

A review of surface theory

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Application: Poincaré's lemma

Theorem

m-dimensional case

If a differential 1-form

m = 2

$$\omega = \sum_{j=1}^{m} \alpha_j(u^1, \dots, u^m)\, du^j$$

$\omega = a\,du + b\,dv$

defined on a simply connected domain $U \subset \mathbb{R}^m$ is closed, that is, $d\omega = 0$ holds, then there exists a C^∞-function f on U such that $df = \omega$. Such a function f is unique up to additive constants.

$\frac{\partial}{\partial u^k}\alpha_j = \frac{\partial}{\partial u^j}\alpha_k$

$df = \sum_{j=1}^{m} \frac{\partial f}{\partial u^j}\, du^j$

commutativity of partial differentials

Application: Conjugation of harmonic functions

Theorem

Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u,v)$ a C^∞-function harmonic on U. Then there exists a C^∞ harmonic function η on U such that $\xi(u,v) + \mathrm{i}\,\eta(u,v)$ is holomorphic on U.

Let $\xi = \xi(u,v)$ be a harmonic function.

$\Delta\xi = \xi_{uu} + \xi_{vv} = 0$

$\Rightarrow$ η is a conjugate harmonic fct

$$\Leftrightarrow \begin{cases} \eta_u = -\xi_v \\ \eta_v = \xi_u \end{cases}$$

Cauchy-Riemann

η : harmonic.

$\Leftrightarrow$ $u + iv \longmapsto \xi + i\eta$ is holomorphic.

- $\eta_u = -\xi_v \quad \eta_v = \xi_u$

$\Leftrightarrow \quad d\eta = -\xi_v\, du + \xi_u\, dv =: \omega$

$d\omega = 0 \Leftrightarrow \xi_{uu} + \xi_{vv} = 0.$

If ξ is harmonic, $\exists\, \eta$ whenever ω is defined on a simply-connected domain.

or

, $\exists\, \eta$ locally.

$\left(\forall p \in U, \ \exists\, V : \text{a nbd of } p, \ \exists\, \eta : \text{on } V \text{ s.t. } d\eta = \omega \right)$

Exercise 3-1

Problem

Let

$\Delta \xi_1 = 0$ $\Delta \xi_2 = 0$

$$\xi_1(u,v) := \frac{u}{u^2+v^2}, \qquad \xi_2(u,v) := \log\sqrt{u^2+v^2}$$

be functions defined on non-simply connected domain $U := \mathbb{R}^2 \setminus \{(0,0)\}$.

1. *Show that both* ξ_1 *and* ξ_2 *are harmonic on* U.
2. *Verify that there exists a conjugate harmonic function* η_1 *of* ξ_1 *on* U.
3. *Prove that there exists no conjugate harmonic function* η_2 *of* ξ_2 *on* U.

global.

($p \in U$ $\exists V \ni p$ = nbd s.t $\exists$ conjugate)

$$\xi_1 = \frac{u}{u^2+v^2} \qquad (\xi_1)_u = \frac{-u^2+v^2}{(u^2+v^2)^2} \qquad (\xi_1)_v = \frac{-2uv}{(u^2+v^2)^2}$$

$$\left(\xi_1 + i\eta = \frac{1}{u+iv}\right)$$

$$(\eta_1)_u = \frac{2uv}{(u^2+v^2)^2} = \frac{\partial}{\partial u}\frac{-v}{u^2+v^2}$$

$$(\eta_1)_v = \frac{-u^2+v^2}{(u^2+v^2)^2}$$

$$U = \mathbb{R}^2 \setminus \{(0,0)\}$$

$$\eta_1 = \frac{-v}{u^2+v^2} + \varphi(v)$$

$$\varphi' = 0$$

$$\eta_1 = \frac{-v}{u^2+v^2} + \text{const}$$

$\xi_2 = \lg\sqrt{u^2+v^2}$ $\qquad$ $\eta_2 = \arg(u+iv)$ $\left(\xi_2 = \operatorname{Re}\lg z,\ z = u+iv\right)$

Assume such $\eta = \eta_2$ exists.

$$d\eta = \eta_u\,du + \eta_v\,dv$$

$$= -\xi_v\,du + \xi_u\,dv = \frac{-v}{u^2+v^2}\,du + \frac{u}{u^2+v^2}\,dv$$

Consider line integral:

along $u = \cos t$, $v = \sin t$

$$\frac{d}{dt}\eta(\cos t, \sin t) = \sin^2 t\,dt + \cos^2 t\,dt$$

$$= dt$$

$$\eta(\cos t, \sin t) - \eta(1, 0) = \int_0^t \eta(\cos t, \sin t)\,dt = t$$

Letting $t = \pi$, $-\pi$

$\eta(-1, 0) = \pi$

$\eta(-1, 0) = -\pi$

contradiction

Exercise 3-2

Problem (Ex. 3-2)

Consider a linear system of partial differential equations for 3×3-matrix valued unknown X on a domain $U \subset \mathbb{R}^2$ as

$$\frac{\partial X}{\partial u} = X\Omega, \quad \frac{\partial X}{\partial v} = X\Lambda,$$

$$\frac{\partial^2 X}{\partial v \partial u} = X_v \Omega + X \Omega_v = X(\Lambda\Omega + \Omega_v)$$

$$\frac{\partial^2 X}{\partial u \partial v} = X(\Omega\Lambda + \Lambda_u)$$

$$\left(\Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix} \right),$$

where (u, v) are the canonical coordinate system of $\mathbb{R}^2$, and α, β and h_j^i $(i, j = 1, 2)$ are smooth functions defined on U. Write down the integrability conditions in terms of α, β and h_j^i.

skew symmetric (i.e. $\Omega^T + \Omega = 0$)

$\Rightarrow (\Omega\Lambda - \Lambda\Omega)^T$: skewsymm.

$$\Omega = \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix},$$

$\overset{?}{=} \Lambda^T\Omega^T - \Omega^T\Lambda^T$

$= \Lambda\Omega - \Omega\Lambda$

$= -(\Omega\Lambda - \Lambda\Omega)$

skew

$$\boxed{\Omega_v - \Lambda_u - \Omega\Lambda + \Lambda\Omega}$$

skew.

$$= \begin{pmatrix} 0 & -\alpha_v + \beta_u + h_1^1 h_2^2 - h_2^1 h_1^2 & -(h_1^1)_v + (h_2^1)_u - \alpha h_2^2 - \beta h_1^2 \\ * & 0 & -(h_1^2)_v + (h_2^2)_u + \alpha h_2^1 - \beta h_1^1 \\ * & * & 0 \end{pmatrix}$$

$\boxed{\text{3-equalities}}$

Our goal: The fundamental theorem of surfaces.

- 1st & 2nd fundamental forms (/w some conditions) determines a surface

Intro geom (200's)

Reference: Umehara & Yamada [UY17]

$U \subset \mathbb{R}^2$: a domain

(not necessarily simply connected)

正则曲面

- $p\colon U \to \mathbb{R}^3$: a regular surface
- $\nu\colon U \to \mathbb{R}^3$: the unit normal vector field.

± ambiguity

$p(u,v) = (x(u,v), y(u,v), z(u,v)) \in \mathbb{R}^3$

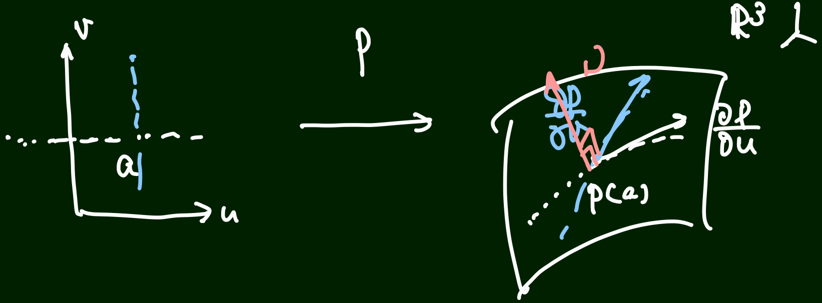
a parametrization of a surface

is a regular surface

def. $\Leftrightarrow$ $p_u = \dfrac{\partial p}{\partial u} = \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix}$, & $p_v = \dfrac{\partial p}{\partial v} = \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix}$

(Examples: Lecture 1)

are linearly independent for each (u,v)



$$\begin{cases} \nu \perp p_u \quad \nu \perp p_v \\ |\nu| = 1 \end{cases}$$

unit normal vector

$\nu = \nu(u, v)$

$dp = p_u du + p_v dv$

1st fundamental form

$dp \cdot dp$

$= (p_u \cdot p_u) du^2$ E

$+ (p_u \cdot p_v) du dv$

$+ (p_v \cdot p_u) dv du$

$+ (p_v \cdot p_v) dv dv$

inner product

$$ds^2 := dp \cdot dp = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

$$\widehat{I} := \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (p_u, p_v),$$

$$II := -d\nu \cdot dp = L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

$$\widehat{II} := \begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (\nu_u, \nu_v)$$

$$-d\nu \cdot dp = -(\nu_u du + \nu_v dv) \cdot (p_u du + p_v dv)$$

$$= -\overset{L}{\nu_u \cdot p_u}\, du^2 - \overset{M}{\nu_u \cdot p_v}\, du\, dv - \overset{M}{\nu_v \cdot p_u}\, dv\, du - \underset{N}{\nu_v \cdot p_v}\, dv^2$$

$\nu_u \cdot p_v$

$= (\nu \cdot p_v)_u - \nu \cdot p_{vu}$

$= -\nu \cdot p_{vu} = -\nu \cdot p_{uv} = \nu_v \cdot p_u$

Curvatures

$\det \hat{I} = |p_u \times p_v|^2 \underset{\neq}{\geq} 0$ (with $p_u \times p_v \neq 0$)

outer product

Weingarten matrix

$$A := \widehat{I}^{-1}\,\widehat{II} = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix},$$

Fact

λ_1, λ_2 : the eigenvalues of A

$\in \mathbb{R}$

- $K := \lambda_1 \lambda_2 = \det A = \dfrac{\det \widehat{II}}{\det \widehat{I}}$: Gaussian curvature
- $H := \dfrac{1}{2}(\lambda_1 + \lambda_2) = \dfrac{1}{2}\operatorname{tr} A.$: Mean curvature.