Advanced Topics in Geometry A1 (MTH.B405)

A review of surface theory

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Application: Poincaré's lemma

Theorem

If a differential 1*-form*

$$\omega = \sum_{j=1}^{m} \alpha_j(u^1, \dots, u^m)\, du^j$$

defined on a simply connected domain $U \subset \mathbb{R}^m$ *is closed, that is,* $d\omega = 0$ *holds, then there exists a* C^∞*-function* f *on* U *such that* $df = \omega$*. Such a function* f *is unique up to additive constants.*

Application: Conjugation of harmonic functions

Theorem

Let $U \subset \mathbb{C} = \mathbb{R}^2$ *be a simply connected domain and* $\xi(u, v)$ *a* C^∞*-function harmonic on* U*. Then there exists a* C^∞ *harmonic function* η *on* U *such that* $\xi(u, v) + \mathrm{i}\,\eta(u, v)$ *is holomorphic on* U*.*

Exercise 3-1

Problem

Let

$$\xi_1(u,v) := \frac{u}{u^2+v^2}, \qquad \xi_2(u,v) := \log\sqrt{u^2+v^2}$$

be functions defined on non-simply connected domain $U := \mathbb{R}^2 \setminus \{(0,0)\}$.

1. *Show that both* ξ_1 *and* ξ_2 *are harmonic on* U.
2. *Verify that there exists a conjugate harmonic function* η_1 *of* ξ_1 *on* U.
3. *Prove that there exists no conjugate harmonic function* η_2 *of* ξ_2 *on* U.

Exercise 3-2

Problem (Ex. 3-2)

Consider a linear system of partial differential equations for 3×3-matrix valued unknown X on a domain $U \subset \mathbb{R}^2$ as

$$\frac{\partial X}{\partial u} = X\Omega, \quad \frac{\partial X}{\partial v} = X\Lambda,$$
$$\left(\Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix} \right),$$

where (u, v) are the canonical coordinate system of $\mathbb{R}^2$, and α, β and h_j^i $(i, j = 1, 2)$ are smooth functions defined on U. Write down the integrability conditions in terms of α, β and h_j^i.

Exercise 3-2

$$\Omega = \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix},$$

$$\begin{aligned} &\Omega_v - \Lambda_u - \Omega\Lambda + \Lambda\Omega \\ &= \begin{pmatrix} 0 & -\alpha_v + \beta_u + h_1^1 h_2^2 - h_2^1 h_1^2 & -(h_1^1)_v + (h_2^1)_u - \alpha h_2^2 - \beta h_1^2 \\ * & 0 & -(h_1^2)_v + (h_2^2)_u + \alpha h_2^1 - \beta h_1^1 \\ * & * & 0 \end{pmatrix} \end{aligned}$$

Immersed surfaces

- $p\colon U \to \mathbb{R}^3$: a regular surface
- $\nu\colon U \to \mathbb{R}^3$: the unit normal vector field.

Fundammental forms

$$
\begin{aligned}
ds^2 &:= dp \cdot dp = E\,du^2 + 2F\,du\,dv + G\,dv^2, \\
\widehat{I} &:= \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (p_u, p_v), \\
II &:= -d\nu \cdot dp == L\,du^2 + 2M\,du\,dv + N\,dv^2, \\
\widehat{II} &:= \begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (\nu_u, \nu_v)
\end{aligned}
$$

Curvatures

$$A := \widehat{I}^{-1}\,\widehat{II} = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix}, \qquad \lambda_1, \lambda_2 \qquad \text{: the eigenvalues of } A$$

$$K := \lambda_1\lambda_2 = \det A = \frac{\det\,\widehat{II}}{\det\,\widehat{I}}$$

$$H := \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}\operatorname{tr} A.$$