Advanced Topics in Geometry A1 (MTH.B405)

A review of surface theory

Kotaro Yamada
kotaro@math.sci.isct.ac.jp
http://www.official.kotaroy.com/class/2025/geom-a1

Institute of Science Tokyo

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"Index" formulation

$p(u,v)$

- $(u^1, u^2) = (u, v)$
- $f_{,i} = \dfrac{\partial f}{\partial u^i}$

(traditional manner : u^i)

$(p_{,1}\,du^1 + p_{,2}\,du^2)\cdot(\;--\;)$

$$ds^2 = dp \cdot dp = \sum_{i,j=1}^{2} g_{ij}\, du^i\, du^j, \qquad (g_{ij} := p_{,i} \cdot p_{,j}), \approx g_{ji}$$

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$$II = -dp \cdot d\nu = \sum_{i,j=1}^{2} h_{ij}\, du^i\, du^j, \quad (h_{ij} := -p_{,i} \cdot \nu_{,j} = -p_{,j} \cdot \nu_{,i})$$

invertible

$$(g^{ij}) := (g_{ij})^{-1} \qquad \sum_k g^{ik} g_{kj} = \delta^i_j$$

= Kronecker's delta

Gauss Frame

- $\mathcal{F}\colon U \ni (u^1,u^2) \mapsto \big(p_{,1}(u^1,u^2), p_{,2}(u^1,u^2), \nu(u^1,u^2)\big) \in \mathrm{GL}(3,\mathbb{R})$

Theorem

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j \qquad \left(\Omega_j := \begin{pmatrix} \Gamma^1_{1j} & \Gamma^1_{2j} & -A^1_j \\ \Gamma^2_{1j} & \Gamma^2_{2j} & -A^2_j \\ h_{1j} & h_{2j} & 0 \end{pmatrix}\right)$$

where

$$\Gamma^k_{ij} := \boxed{\frac{1}{2}\sum_{l=1}^{2} g^{kl}(g_{il,j}+g_{lj,i}-g_{ij,l})} \qquad (i,j,k=1,2)$$

2nd fund. form

Thm

- $p_{,11} = \Gamma^1_{11}\, p_{,1} + \Gamma^2_{11}\, p_{,2} + h_{11}\,\nu$
- $\nu_{,1} = -A^1_1\, p_{,1} - A^2_1\, p_{,2} + 0\,\nu$

Weingarten matrix

lin indep

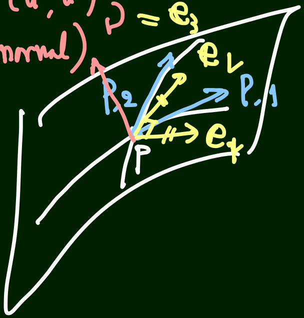
$$\mathfrak{F} = \underbrace{\left(P_{,1} \quad P_{,2} \quad \nu \right)}_{3\times 3 \text{ matrix}} \in GL(3, \mathbb{R})$$

$(\det \neq 0)$

$= \{ 3\times 3$ real invertible matrices $\}$

Gauss Frame

- basis of $\mathbb{R}^3$
- depends on (u^1, u^2)
- (tang tang normal)



Adapted Frame

- $p\colon U \to \mathbb{R}^3$: a regular surface
- ν: the unit normal
- $\mathcal{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) : U \to \mathrm{SO}(3)$, $(\boldsymbol{e}_3 = \nu)$: an adapted frame

$$\check{I} = \begin{pmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^2 \end{pmatrix}, \qquad \text{such that} \qquad (p_u, p_v) = (\boldsymbol{e}_1, \boldsymbol{e}_2)\,\check{I}$$

$$\check{II} = \begin{pmatrix} h_1^1 & h_2^1 \\ h_1^2 & h_2^2 \end{pmatrix} \qquad \text{such that} \qquad ((\boldsymbol{e}_3)_u, (\boldsymbol{e}_3)_v) = -(\boldsymbol{e}_1, \boldsymbol{e}_2)\,\check{II}\,.$$

$(e_3)_u$: tangent

$(e_3)_u \cdot e_3 = \frac{1}{2}(\underbrace{e_3 \cdot e_3}_{1})_u = 0$

i.e. $(e_3)_u \perp e_3$

Gauss-Weingarten formula

skew symmetric.

$\because$ $\mathcal{E}$ is $SO(3)$-valued

{skew symm matrices} is the Lie algebra of $SO(3)$

$$\mathcal{E}_u = \mathcal{E}\Omega, \qquad \mathcal{E}_v = \mathcal{E}\Lambda$$

$$\left(\Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix}\right).$$

$$0 = (\mathrm{id})_u = (\mathcal{E}\mathcal{E}^T)_u = \mathcal{E}_u\mathcal{E}^T + \mathcal{E}\mathcal{E}_u^T$$

$$= \mathcal{E}\Omega\mathcal{E}^T + \mathcal{E}(\mathcal{E}\Omega)^T$$

$$= \mathcal{E}\Omega\mathcal{E}^T + \mathcal{E}\Omega^T\mathcal{E}^T = \mathcal{E}\underbrace{(\Omega + \Omega^T)}_{0}\mathcal{E}^T$$

regular

Exercise 4-1

$|p_u|^2 = |p_v|^2 = e^{2\sigma}$

$p_u \perp p_v$

Problem (Ex. 4-1)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ *are given in the form*

$g_{11} = g_{22} = e^{2\sigma},\ g_{12} = g_{21} = 0$

$$ds^2 = e^{2\sigma}((du^1)^2 + (du^2)^2), \qquad II = \sum_{i,j=1}^{2} h_{ij}\, du^i\, du^j,$$

where σ *is a smooth function in* (u^1, u^2).

$P^k_l \quad A^j_i$

1. *Compute the matrices* Ω_j $(j = 1, 2)$ *in* (4.17).
2. *Set* $(u, v) = (u_1, u_2)$, $\boldsymbol{e}_1 := e^{-\sigma} p_{u_1}$, $\boldsymbol{e}_2 := e^{-\sigma} p_{u_2}$, *and* $\boldsymbol{e}_3 = \nu$, *where* ν *is the unit normal vector field. Compute the matrices* Ω *and* Λ *in* (4.31) *for the orthonormal frame* $\mathcal{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$.

Exercise 4-2

$K = \dfrac{\det \widehat{II}}{\det \hat{I}} = \det\begin{pmatrix} 0 & \sin\theta \\ \sin\theta & 0 \end{pmatrix} \Big/ \det\begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}$

$= -1$

Problem (Ex. 3-2)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ are given in the form

$$ds^2 = (du^1)^2 + 2\cos\theta\, du^1\, du^2 + (du^2)^2, \qquad II = 2\sin\theta\, du^1\, du^2,$$

where θ is a smooth function in (u^1, u^2).

1. *Compute the matrices Ω_j $(j = 1, 2)$ in (4.17).*
2. *Find an adapted frame, and compute the matrices Ω and Λ in (4.31).* (e_1, e_2)

→ 2Q, pseudospherical surfaces.