

Advanced Topics in Geometry A1 (MTH.B405)

A review of surface theory

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“Index” formulation

- $(u^1, u^2) = (u, v)$
- $f_{,i} = \frac{\partial f}{\partial u^i}$

$$ds^2 = dp \cdot dp = \sum_{i,j=1}^2 g_{ij} du^i du^j, \quad (g_{ij} := p_{,i} \cdot p_{,j}),$$

$$II = -dp \cdot d\nu = \sum_{i,j=1}^2 h_{ij} du^i du^j, \quad (h_{ij} := -p_{,i} \cdot \nu_{,j} = -p_{,j} \cdot \nu_{,i})$$

$$(g^{ij}) := (g_{ij})^{-1} \quad \sum_k g^{ik} g_{kj} = \delta_j^i$$

Gauss Frame

$$\mathcal{F}: U \ni (u^1, u^2) \mapsto (p_{,1}(u^1, u^2), p_{,2}(u^1, u^2), \nu(u^1, u^2)) \in \mathrm{GL}(3, \mathbb{R})$$

Theorem

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F} \Omega_j \quad \left(\Omega_j := \begin{pmatrix} \Gamma_{1j}^1 & \Gamma_{2j}^1 & -A_j^1 \\ \Gamma_{1j}^2 & \Gamma_{2j}^2 & -A_j^2 \\ h_{1j} & h_{2j} & 0 \end{pmatrix} \right)$$

where

$$\Gamma_{ij}^k := \frac{1}{2} \sum_{l=1}^2 g^{kl} (g_{il,j} + g_{lj,i} - g_{ij,l}), \quad (i, j, k = 1, 2)$$

Adapted Frame

- $p: U \rightarrow \mathbb{R}^3$: a regular surface
- ν : the unit normal
- $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) : U \rightarrow \text{SO}(3)$, $(\mathbf{e}_3 = \nu)$: an adapted frame

$$\check{I} = \begin{pmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^2 \end{pmatrix} \quad \text{such that} \quad (p_u, p_v) = (\mathbf{e}_1, \mathbf{e}_2) \check{I}$$

$$\check{II} = \begin{pmatrix} h_1^1 & h_2^1 \\ h_1^2 & h_2^2 \end{pmatrix} \quad \text{such that} \quad ((\mathbf{e}_3)_u, (\mathbf{e}_3)_v) = -(\mathbf{e}_1, \mathbf{e}_2) \check{II}.$$

Gauss-Weingarten formula

$$\mathcal{E}_u = \mathcal{E}\Omega, \quad \mathcal{E}_v = \mathcal{E}\Lambda$$
$$\left(\Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix} \right).$$

Exercise 4-1

Problem (Ex. 4-1)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ are given in the form

$$ds^2 = e^{2\sigma}((du^1)^2 + (du^2)^2), \quad II = \sum_{i,j=1}^2 h_{ij} du^i du^j,$$

where σ is a smooth function in (u^1, u^2) .

- ❶ *Compute the matrices Ω_j ($j = 1, 2$) in (4.17).*
- ❷ *Set $(u, v) = (u_1, u_2)$, $e_1 := e^{-\sigma} p_{u_1}$, $e_2 := e^{-\sigma} p_{u_2}$, and $e_3 = \nu$, where ν is the unit normal vector field. Compute the matrices Ω and Λ in (4.31) for the orthonormal frame $\mathcal{E} = (e_1, e_2, e_3)$.*

Exercise 4-2

Problem (Ex. 3-2)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ are given in the form

$$ds^2 = (du^1)^2 + 2 \cos \theta du^1 du^2 + (du^2)^2, \quad II = 2 \sin \theta du^1 du^2,$$

where θ is a smooth function in (u^1, u^2) .

- ❶ *Compute the matrices Ω_j ($j = 1, 2$) in (4.17).*
- ❷ *Find an adapted frame, and compute the matrices Ω and Λ in (4.31).*