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Info. Sheet 4; Advanced Topics in Geometry A1 (MTH.B405)

Corrections

- Lecture Note, page 13, line 14 and page 15, line 15: [?] ⇒ [Lee13] (see the bibliography on the LMS).
- Lecture Note, page 15, line 2: $\sigma \Rightarrow \sigma_0$
- Lecture Note, page 17, footnote 8: differentiablity ⇒ differentiability
- Lecture Note, page 18, the 3rd line of Exercise 3-1: be a function ⇒ functions
- Blackboard C and Handout C, page 10: Proof of Theorem 2.5 ⇒ Proof of Theorem 3.5
- Blackboard C and Handout C, page 10: functaion ⇒ function (2 times)

Students' comments

- I think the course is good. No request this time.

  Lecturer's comment I see.

- 今日もありがとうございました．Thank you for today.

  Lecturer's comment どういたしまして．You're welcome.

Q and A

Q 1: 1-form α を $\alpha = -\xi_v\,du + \xi_u\,dv$ とおくことで，α が closed と ξ が調和，α が exact と ξ が共役調和関数をもつ，がそれぞれ対応すると理解しました．U が単連結なら closed ⇒ exact がいえますが，この仮定は必要十分なのでしょうか．もう少し弱めることができますか？ また 3-1 で同じ領域でも exact になるかどうかに差がでていますが，exact になるような ξ の方の条件についてわかっていることはありますか．

I understood that, by setting the 1-form α as $\alpha = -\xi_v\,du + \xi_u\,dv$, closedness of α and exactness of α correspond to harmonicity and existence of the conjugate harmonic function, respectively. If U is simply connected, then "closed ⇒ exact" holds, but is this assumption necessary and sufficient? Can it be weakened a little? Also, in 3-1, there is a difference in whether the same region is exact or not. What do we know about the condition for ξ to be exact?

A: A closed 1-form can be exact even if U is not simply connected, as you see in Problem 3-1, (2). The condition for a closed one form to be exact is not simple, e.g., it depends on the cohomology of the region, and in the case of problem 3-1, it is related to the residue of the holomorphic function at the isolated singularity.

Q 2: U 上関数の conjugate harmonic function の存在性は 1-form ω に対して $df = \omega$ となる C^∞ 関数 f があるかどうかに帰着されますが，一方で複素関数の議論もできそうです．単連結とは限らない領域 U 上の関数に conjugate harmonic function が存在するか判定する方法は知られていますか．

Existence of the conjugate harmonic function of a function on a domain U is reduced to existence of a smooth function f satisfying $df = \omega$. On the other hand, it could be argued with complex function. Is there any known way to determine whether there exists a conjugate harmonic function for a function on the domain U that is not necessarily simply connected?

A: Depending on a topology of the domain (cf. de Rham cohomology). For the case as in Problem 3-1, it could be related to the residue of holomorphic functions.

Q 3: Exercise 3-1 で ξ_1 が U 上共役な調和関数が存在して，ξ_2 が U 上共役な調和関数が存在しないのは，ξ_i を実部にもつ複素正則関数が原点に解析接続できるかが鍵になっているのでしょうか．

In Exercise 3-1, there exists a conjugate harmonic function of ξ_1 on U and not for ξ_2. The key to this phenomena is the possibility of analytic continuation of a holomorphic function with ξ_i as real part to the origin?

A: Yes, the analytic continuation <u>around</u> the origin.

Q 4: Thm 3.9 では単連結領域上の調和関数には conjugate な調和関数が存在するとなっていますが，今回の Exercise 3-1 のような非単連結な領域で conjugate が存在する十分条件は何があるのでしょうか.
Thm 3.9 states that there exists a conjugate harmonic functions on a simply connected domain, but what are the sufficient conditions for existence of conjugate harmonic function a non simply connected domain such as in Exercise 3-1?

A: No, in general. It is related to both the topology of the domain and the property of function.

Q 5: Though I understand the proof of Poincaré's lemma, there is a formulation of sentence that is strange to me. When in the last paragraph it is written that "Proposition 2.8 yields $\xi = \det \xi$ never vanishes", the important point that we are keeping from Proposition 2.8 in this case is not that $\xi = \det \xi$ right? Since it is a well-known fact, the important thing that Proposition 2.8 allows us to exploit even in the case of "1×1 matrices" is that $\xi = (\det \xi =)\xi(0) \exp \int_{t_0}^{t} \alpha(\tau)\, d\tau$ and thus is always of same sign as $\xi(0)$. This sentence makes me wonder if my understanding is correct?

A: It is correct.

Q 6: Is there a connection between differential forms and homology? Because the d operator looks like the boundary operator in homology ($d^2 = 0$).

A: Actually, differential forms and d-operator determines the *cohomology* called *de Rham cohomology*. If you are interested, check the word "de Rham's theorem" in differential geometry.

Q 7: I have some doubts about Theorem (Poincaré's lemma). What is the meaning of $d\omega = 0$? And also, what is the meaning of "unique up to additive constants". Does this mean that the form of such function is: $f + c$(constant) which satisfies $df = \omega$?

A: When $\omega = a\, du + b\, dv$, where a and b are functions in (u, v), $d\omega = (b_v - a_u)\, du \wedge dv$, by definition. Accordingly, $d\omega = 0$ means $b_v - a_u = 0$. Regarding the phrase "unique up to additive constants", your understanding is correct. In other words, "If both f and g are functions satisfying $df = dg = \omega$, there exists a constant c such that $g = f + c$".