Advanced Topics in Geometry A1 (MTH.B405)

Gauss and Codazzi equations

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The Gauss-Weingarten formulas

$p = p(u^1, u^2)$: a parametrized surface

$\nu = \nu(u^1, u^2)$: the unit normal vector field

$\mathcal{F} = (p_{,1}, p_{,2}, \nu)$: the Gauss Frame   3 × 3

The Gauss-Weingarten formula:

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j \qquad (j = 1, 2)$$

The Gauss-Weingarten formulas

$\mathcal{F} = (p_{,1}, p_{,2}, \nu)$

$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j$

(e.g.) $p_{,11} = \Gamma^1_{j1} p_{,1} + \Gamma^2_{j1} p_{,2} + h_{j1}\nu$

$$\Omega_j = \begin{pmatrix} \Gamma^1_{j1} & \Gamma^1_{j2} & -A^1_j \\ \Gamma^2_{j1} & \Gamma^2_{j2} & -A^2_j \\ h_{j1} & h_{j2} & 0 \end{pmatrix}$$

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}(g_{lj,i} + g_{il,j} - g_{ij,l})$$

Christoffel's symbol

$$g_{ij} = p_{,i} \cdot p_{,j}, \qquad (g^{ij}) = (g_{ij})^{-1}$$

$$h_{ij} = -p_{,i} \cdot \nu_{,j} = -\nu_{,i} \cdot p_{,j}$$

2nd fund form

$$A^i_j = \sum_l g^{il} h_{lj}$$

Exercise 4-1

Problem (Ex. 4-1)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ *are given in the form* $ds^2 = \sum g_{ij}\, du^i du^j$

$$\underline{ds^2 = e^{2\sigma}((du^1)^2 + (du^2)^2)}, \qquad II = \sum_{i,j=1}^{2} \underline{h_{ij}}\, du^i\, du^j,$$

where σ *is a smooth function in* (u^1, u^2).

- *Compute the matrices* Ω_j $(j = 1, 2)$.

$g_{11} = g_{22} = e^{2\sigma}$ $\qquad g_{12} = g_{21} = 0$

$\hat{I} = e^{2\sigma}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad \hat{I}^{-1} = (g^{ij}) = e^{-2\sigma}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Exercise 4-1-1

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} e^{2\sigma} & 0 \\ 0 & e^{2\sigma} \end{pmatrix}, \qquad \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} e^{-2\sigma} & 0 \\ 0 & e^{-2\sigma} \end{pmatrix},$$

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^{2} g^{kl} (g_{il,j} + g_{lj,i} - g_{ij,l})$$

$$\Gamma^1_{11} = \frac{1}{2} \sum_{l=1}^{2} g^{1l} (g_{1l,1} + g_{l1,1} - g_{11,l})$$

$$= \frac{1}{2} g^{11} (g_{11,1} + \cancel{g_{11,1}} - \cancel{g_{11,1}})$$

$$= \frac{1}{2} g^{11} (g_{11,1}) = \frac{1}{2} e^{-2\sigma} (e^{2\sigma})_{,1} = \sigma_{,1}$$

$$
\begin{cases}
\Gamma^1_{11} = \sigma_{,1} & \Gamma^2_{11} = -\sigma_{,2} \\
\Gamma^1_{12} = \Gamma^1_{21} = \sigma_{,2} & \Gamma^2_{12} = \Gamma^2_{21} = \sigma_{,1} \\
\Gamma^1_{22} = -\sigma_{,1} & \Gamma^2_{22} = \sigma_{,2}
\end{cases}
$$

Exercise 4-1-1

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} e^{-2\sigma} & 0 \\ 0 & e^{-2\sigma} \end{pmatrix}, \qquad A = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

$$A = \mathrm{I}^{-1}\,\mathrm{II} = e^{-2\sigma} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

$$A^i_j = e^{-2\sigma} h_{ij}$$

Exercise 4-1-1

$$\Omega_1 = \begin{pmatrix} \sigma_{,1} & \sigma_{,2} & -e^{-2\sigma}h_{11} \\ -\sigma_{,2} & \sigma_{,1} & -e^{-2\sigma}h_{21} \\ h_{11} & h_{21} & 0 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} \sigma_{,2} & -\sigma_{,1} & -e^{-2\sigma}h_{12} \\ \sigma_{,1} & \sigma_{,2} & -e^{-2\sigma}h_{22} \\ h_{12} & h_{22} & 0 \end{pmatrix}.$$

$$\frac{\partial \mathcal{F}}{\partial u^1} = \mathcal{F}\Omega_1, \quad \frac{\partial \mathcal{F}}{\partial u^2} = \mathcal{F}\Omega_2$$

G-W equations

for $\underline{ds^2 = e^{2\sigma}\left((du^1)^2 + (du^2)^2\right)}$

(u^1, u^2): isothermal parameters
conformal parameters.

Exercise 4-1-2

Problem (Ex. 4-1)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ *are given in the form*

$(u,v) = (u^1, u^2)$

$$ds^2 = e^{2\sigma}((du^1)^2 + (du^2)^2), \qquad II = \sum_{i,j=1}^{2} h_{ij}\, du^i\, du^j,$$

$P,1$

where σ *is a smooth function in* (u^1, u^2).

- *Set* $(u, v) = (u^1, u^2)$, $\boxed{\boldsymbol{e}_1 := e^{-\sigma} p_{u^1}}$, $\boxed{\boldsymbol{e}_2 := e^{-\sigma} p_{u^2}}$, *and* $\boldsymbol{e}_3 = \nu$, *where* ν *is the unit normal vector field. Compute the matrices* Ω *and* Λ *for the orthonormal frame* $\mathcal{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$.

$$e_1 = e^{-\sigma} P_u , \quad e_2 = e^{-2\sigma} P_v , \quad e_3 = \nu$$

- $P_u \cdot P_u = g_{11} = e^{2\sigma} = P_v \cdot P_v$, $P_u \cdot P_v = 0$

$\Rightarrow$ $|e_1| = |e_2| = 1$ $\quad e_1 \perp e_2$; $\quad e_j \cdot e_3 = 0$ $\quad j = 1, 2$

$|e_3| = 1$

★ $\mathcal{E} = (e_1, e_2, \overset{\nu}{e_3})$: orthonormal frame

adapted frame.

$$\frac{\partial \mathcal{E}}{\partial u} = \mathcal{E}\,\underline{\underline{\Omega}} \qquad \frac{\partial \mathcal{E}}{\partial v} = \mathcal{E}\,\underline{\underline{\Lambda}}$$

$$P_{uu} = \Gamma^1_{11} P_u + \Gamma^2_{11} P_v + h_{11}\nu = \underline{\sigma_u P_u - \sigma_v P_v + h_{11}\nu}$$

$$P_{vu} = \sigma_2 P_u + \sigma_1 P_v + h_{21}\nu$$

$$P_{vv} = -\sigma_1 P_u + \sigma_2 P_v + h_{22}\nu$$

$$\nu_u = -e^{-2\sigma} h_{11} P_u - e^{-2\sigma} h_{21} P_v \;\cdots$$

$$(e_1)_u = (e^{-\sigma} P_u)_u = -\sigma_u e^{-\sigma} P_u + e^{-\sigma} P_{uu}$$

$$= \cancel{-\sigma_u e^{-\sigma} P_u} + \cancel{e^{-\sigma} \sigma_u P_u} + \underline{e}^{-\sigma} \sigma_v \underline{P_v} + e^{-\sigma} h_{11}\nu$$

$$= +\sigma_v e_2 + e^{-\sigma} h_{11} e_3 \quad \cdots$$

$$(e_3)_u = \nu_u = -h_{11} e^{-\sigma} e_1 - h_{12} e^{-\sigma} e_2$$

Exercise 4-1-2

$\mathcal{E} = (\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3)$

7th lecture.

$$\frac{\partial \mathcal{E}}{\partial u} = \mathcal{E}\,\Omega$$

$$\Omega = \begin{pmatrix} 0 & -\sigma_v & -e^{-\sigma}h_{11} \\ \sigma_v & 0 & -e^{-\sigma}h_{21} \\ e^{-\sigma}h_{11} & e^{-\sigma}h_{21} & 0 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 0 & -\sigma_u & -e^{-\sigma}h_{12} \\ \sigma_u & 0 & -e^{-\sigma}h_{22} \\ e^{-\sigma}h_{12} & e^{-\sigma}h_{22} & 0 \end{pmatrix}$$

skew-symmetric

- $\mathcal{E}$: orthogonal matrices valued

$\Rightarrow$ $\mathcal{E}^{-1}\dfrac{\partial \mathcal{E}}{\partial u}$: skew-symmetric

Exercise 4-2

$2\cos\theta\, du^1 du^2$
$= \cos\theta\, du^1 du^2 + \cos\theta\, du^2 du^1$

$g_{12} = g_{21}$

Problem (Ex. 4-2)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ *are given in the form*

$$ds^2 = (du^1)^2 + 2\cos\theta\, du^1\, du^2 + (du^2)^2, \qquad II = 2\sin\theta\, du^1\, du^2,$$

where θ *is a smooth function in* (u^1, u^2). *Compute the matrices* Ω_j $(j = 1, 2)$

$\theta \in (0, \pi)$

$$\hat{I} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}$$

$\det \hat{I} = 1 - \cos^2\theta = \sin^2\theta$

$= 0$?

p: not regular surface.

singularity.

$\boxed{\text{Asymptotic Chebyshev net}}$

$\to 2Q$

Recall

$\csc\theta = \operatorname{cosec}\theta = \frac{1}{\sin\theta}$

$\cot\theta = \frac{1}{\tan\theta}$

K = Gaussian curvature

$= \frac{\det \hat{\mathrm{II}}}{\det \hat{\mathrm{I}}} = -1$

$$g_{11} = g_{22} = 1, \qquad g_{12} = g_{21} = \cos\theta,$$
$$h_{11} = h_{22} = 0, \qquad h_{12} = h_{21} = \sin\theta$$

$$\hat{\mathrm{I}} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix} \qquad \hat{\mathrm{I}}^{-1} = \frac{1}{\sin^2\theta}\begin{pmatrix} 1 & -\cos\theta \\ -\cos\theta & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \csc^2\theta & -\csc\theta\cot\theta \\ -\csc\theta\cot\theta & \csc^2\theta \end{pmatrix}$$

$$\hat{\mathrm{II}} = \sin\theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A = \hat{\mathrm{I}}^{-1}\hat{\mathrm{II}} = \begin{pmatrix} -\cot\theta & \csc\theta \\ \csc\theta & -\cot\theta \end{pmatrix}$$

Exercise 4-2

$\theta \in (0, \pi)$

$$\Omega_1 = \begin{pmatrix} \theta_{,1}\cot\theta & 0 & \cot\theta \\ -\theta_{,1}\csc\theta & 0 & -\csc\theta \\ 0 & \sin\theta & 0 \end{pmatrix}$$

$$\Omega_2 = \begin{pmatrix} 0 & -\theta_{,2}\csc\theta & -\csc\theta \\ 0 & \theta_{,2}\cot\theta & \cot\theta \\ \sin\theta & 0 & 0 \end{pmatrix}$$

When $\theta \equiv 0 \pmod{\pi}$ singularity.

$\Rightarrow$ $\csc\theta$, $\cot\theta$: diverges

Orthonormal frame?