Advanced Topics in Geometry A1 (MTH.B405)

Gauss and Codazzi equations

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The Gauss-Weingarten formulas

$$
\begin{aligned}
p = p(u^1, u^2) &: \text{a parametrized surface} \\
\nu = \nu(u^1, u^2) &: \text{the unit normal vector field} \\
\mathcal{F} = (p_{,1}, p_{,2}, \nu) &: \text{the Gauss Frame}
\end{aligned}
$$

The Gauss-Weingarten formula:

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j \qquad (j = 1, 2)$$

The Gauss-Weingarten formulas

$$\Omega_j = \begin{pmatrix} \Gamma^1_{j1} & \Gamma^1_{j2} & -A^1_j \\ \Gamma^2_{j1} & \Gamma^2_{j2} & -A^2_j \\ h_{j1} & h_{j2} & 0 \end{pmatrix}$$

$$\Gamma^k_{ij} = \frac{1}{2} \sum_l g^{kl}(g_{lj,i} + g_{il,j} - g_{ij,l})$$

$$g_{ij} = p_{,i} \cdot p_{,j}, \qquad (g^{ij}) = (g_{ij})^{-1}$$

$$h_{ij} = -p_{,i} \cdot \nu_{,j} = -\nu_{,i} \cdot p_{,j}$$

$$A^i_j = \sum_l g^{il} h_{lj}$$

Exercise 4-1

Problem (Ex. 4-1)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ *are given in the form*

$$ds^2 = e^{2\sigma}((du^1)^2 + (du^2)^2), \qquad II = \sum_{i,j=1}^{2} h_{ij}\, du^i\, du^j,$$

where σ *is a smooth function in* (u^1, u^2)*.*

- *Compute the matrices* Ω_j $(j = 1, 2)$*.*

Exercise 4-1-1

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} e^{2\sigma} & 0 \\ 0 & e^{2\sigma} \end{pmatrix}, \qquad \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} e^{-2\sigma} & 0 \\ 0 & e^{-2\sigma} \end{pmatrix},$$

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^{2} g^{kl}(g_{il,j} + g_{lj,i} - g_{ij,l})$$

Exercise 4-1-1

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} e^{-2\sigma} & 0 \\ 0 & e^{-2\sigma} \end{pmatrix}, \qquad A = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

Exercise 4-1-1

$$\Omega_1 = \begin{pmatrix} \sigma_{,1} & \sigma_{,2} & -e^{-2\sigma} h_{11} \\ -\sigma_{,2} & \sigma_{,1} & -e^{-2\sigma} h_{21} \\ h_{11} & h_{21} & 0 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} \sigma_{,2} & -\sigma_{,1} & -e^{-2\sigma} h_{12} \\ \sigma_{,1} & \sigma_{,2} & -e^{-2\sigma} h_{22} \\ h_{12} & h_{22} & 0 \end{pmatrix}.$$

Exercise 4-1-2

Problem (Ex. 4-1)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ *are given in the form*

$$ds^2 = e^{2\sigma}((du^1)^2 + (du^2)^2), \qquad II = \sum_{i,j=1}^{2} h_{ij}\, du^i\, du^j,$$

where σ *is a smooth function in* (u^1, u^2)*.*

- *Set* $(u, v) = (u^1, u^2)$*,* $\boldsymbol{e}_1 := e^{-\sigma} p_{u^1}$*,* $\boldsymbol{e}_2 := e^{-\sigma} p_{u^2}$*, and* $\boldsymbol{e}_3 = \nu$*, where* ν *is the unit normal vector field. Compute the matrices* Ω *and* Λ *for the orthonormal frame* $\mathcal{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$*.*

Exercise 4-1-2

$$\Omega = \begin{pmatrix} 0 & -\sigma_v & -e^{-\sigma}h_{11} \\ \sigma_v & 0 & -e^{-\sigma}h_{21} \\ e^{-\sigma}h_{11} & e^{-\sigma}h_{21} & 0 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} 0 & -\sigma_u & -e^{-\sigma}h_{12} \\ \sigma_u & 0 & -e^{-\sigma}h_{22} \\ e^{-\sigma}h_{12} & e^{-\sigma}h_{22} & 0 \end{pmatrix}$$

Exercise 4-2

Problem (Ex. 4-2)

Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ *are given in the form*

$$ds^2 = (du^1)^2 + 2\cos\theta\, du^1\, du^2 + (du^2)^2, \qquad II = 2\sin\theta\, du^1\, du^2,$$

where θ *is a smooth function in* (u^1, u^2). *Compute the matrices* Ω_j $(j = 1, 2)$

Exercise 4-2

$$g_{11} = g_{22} = 1, \qquad g_{12} = g_{21} = \cos\theta,$$
$$h_{11} = h_{22} = 0, \qquad h_{12} = h_{21} = \sin\theta$$

Exercise 4-2

$$\Omega_1 = \begin{pmatrix} \theta_{,1} \cot\theta & 0 & \cot\theta \\ -\theta_{,1} \csc\theta & 0 & -\csc\theta \\ 0 & \sin\theta & 0 \end{pmatrix}$$

$$\Omega_2 = \begin{pmatrix} 0 & -\theta_{,2} \csc\theta & -\csc\theta \\ 0 & \theta_{,2} \cot\theta & \cot\theta \\ \sin\theta & 0 & 0 \end{pmatrix}$$