Advanced Topics in Geometry A1 (MTH.B405)

Gauss and Codazzi equations

Kotaro Yamada
kotaro@math.sci.isct.ac.jp
http://www.official.kotaroy.com/class/2025/geom-a1

Institute of Science Tokyo

2025/05/23

Exercise 4-2, continued

Asymptotic Chebyshev net

$$(u, v) = (u^1, u^2)$$

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2, II = 2\sin\theta\, du\, dv$$

$$\widehat{I} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}, \qquad \widehat{II} = \begin{pmatrix} 0 & \sin\theta \\ \sin\theta & 0 \end{pmatrix}$$

$$(P_u + P_v)\cdot(P_u - P_v) = P_u \cdot P_u - P_v \cdot P_v = 1 - 1 = 0$$

$$(P_u + P_v)\cdot(P_u + P_v) = \overset{1}{P_u \cdot P_u} + 2\overset{\cos\theta}{P_u \cdot P_v} + \overset{1}{P_v \cdot P_v}$$

$$= 2(1 + \cos\theta) = 4\cos^2\frac{\theta}{2}$$

$$(P_u - P_v)\cdot(P_u - P_v) = 2(1 - \cos\theta) = 4\sin^2\frac{\theta}{2}$$

$$e_1 := \frac{1}{2}\sec\frac{\theta}{2}(P_u + P_v), \quad e_2 = \frac{1}{2}\csc\frac{\theta}{2}(P_u - P_v)$$

$e_3 := \nu \qquad \mathcal{E} = (e_1\ e_2\ e_3)$: adapted.

Q and A

Q: My question is actually about the exercise 4-2. Is it normal that there is a possibility for $\det \widehat{I}$ vanish? Wouldn't that be contradictory to the definition of p as a parametrization according to the Cauchy-Schuwarz inequality as exploited in (4.8).

- $\det \hat{I}$ vanish $\Rightarrow$ Gauss & Weingarten do not work (singularity)
-

$\mathcal{E} = (\boldsymbol{e}_1 \ \boldsymbol{e}_2 \ \boldsymbol{e}_3)$

$\boldsymbol{e}_1 = \frac{1}{2}\sec\frac{\theta}{2}(p_u + p_v) \quad \boldsymbol{e}_2 = \frac{1}{2}\csc\frac{\theta}{2}(p_u - p_v)$

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2,\, II = 2\sin\theta\, du\, dv$$

$$p_u \cdot p_u = p_v \cdot p_v = 1, \qquad p_u \cdot p_v = \cos\theta$$

$$p_u = \cos\frac{\theta}{2}\boldsymbol{e}_1 + \sin\frac{\theta}{2}\boldsymbol{e}_2, \quad p_v = \cos\frac{\theta}{2}\boldsymbol{e}_1 - \sin\frac{\theta}{2}\boldsymbol{e}_2, \quad \nu = \boldsymbol{e}_3$$

$$\frac{\partial \mathcal{E}}{\partial u} = \mathcal{E}\Omega = \mathcal{E}\begin{pmatrix} 0 & -p & -q \\ p & 0 & -r \\ q & r & 0 \end{pmatrix}$$

$$\frac{\partial \boldsymbol{e}_1}{\partial u} = p\boldsymbol{e}_2 + q\boldsymbol{e}_3 \qquad \frac{\partial \boldsymbol{e}_2}{\partial u} = -p\boldsymbol{e}_1 + r\boldsymbol{e}_3$$

$$P_u = \cos\frac{\theta}{2}\,\Theta_1 + \sin\frac{\theta}{2}\,\Theta_2$$

$$+\sin\frac{\theta}{2}(\Theta_1)_u$$

$$P_{uu} = \frac{\theta_u}{2}\left(-\sin\frac{\theta}{2}\,\Theta_1 + \cos\frac{\theta}{2}\,\Theta_2\right) + \cos\frac{\theta}{2}(\Theta_1)_u$$

$$\overset{GW}{=} \theta_u\left(\cot\theta\, P_u - \csc\theta\, P_v\right)$$

$$= \theta_u\Big(\left(\cot\theta\cos\frac{\theta}{2} - \csc\theta\cos\frac{\theta}{2}\right)\Theta_1$$

$$+\left(\cot\theta\cos\frac{\theta}{2} + \csc\theta\sin\frac{\theta}{2}\right)\Theta_2\Big)$$

$$= \theta_u\left(-\sin\frac{\theta}{2}\,\Theta_1 + \cos\frac{\theta}{2}\,\Theta_2\right) \quad \cdots$$

$$\cot\theta - \csc\theta = \frac{\cos\theta - 1}{\sin\theta} = \frac{-2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -2\tan\frac{\theta}{2}$$

Exercise 4-2, continued

$$p_u = \cos\frac{\theta}{2}\boldsymbol{e}_1 + \sin\frac{\theta}{2}\boldsymbol{e}_2, \quad p_v = \cos\frac{\theta}{2}\boldsymbol{e}_1 - \sin\frac{\theta}{2}\boldsymbol{e}_2, \quad \nu = \boldsymbol{e}_3$$

$$\mathcal{E} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3); \qquad \mathcal{E}_u = \mathcal{E}\Omega, \quad \mathcal{E}_v = \mathcal{E}\Lambda$$

$$\Omega = \begin{pmatrix} 0 & -\frac{\theta_u}{2} & -\sin\frac{\theta}{2} \\ \frac{\theta_u}{2} & 0 & \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & \frac{\theta_v}{2} & -\sin\frac{\theta}{2} \\ -\frac{\theta_v}{2} & 0 & -\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 \end{pmatrix},$$

- works even if $\theta = 0, \pi$.
- $\to$ 2Q

The ingetrablility conditions

$\mathcal{F} = (P_{,1}\ P_{,2}\ \nu)$

The Gauss-Weingarten formulas:

- $$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j, \qquad \Omega_j = \begin{pmatrix} \Gamma^1_{j1} & \Gamma^1_{j2} & -A^1_j \\ \Gamma^2_{j1} & \Gamma^2_{j2} & -A^2_j \\ h_{j1} & h_{j2} & 0 \end{pmatrix} \qquad (j = 1, 2)$$

The integrability conditions:

If $\exists\ \mathcal{F}$

$$\Rightarrow \boxed{\frac{\partial \Omega_1}{\partial u^2} - \frac{\partial \Omega_2}{\partial u^1} - \Omega_1\Omega_2 + \Omega_2\Omega_1 = O}$$

$$\frac{\partial^2 \mathcal{F}}{\partial u^2 \partial u^1} = \frac{\partial^2 \mathcal{F}}{\partial u^1 \partial u^2}$$

$$= \frac{\partial}{\partial u^2}\left(\frac{\partial \mathcal{F}}{\partial u^1}\right) = \frac{\partial}{\partial u^2}(\mathcal{F}\Omega_1) = \frac{\partial \mathcal{F}}{\partial u^2}\Omega_1 + \mathcal{F}\frac{\partial \Omega_1}{\partial u^2}$$

$$= \mathcal{F}\left(\Omega_2\Omega_1 + \frac{\partial \Omega_2}{\partial u^1}\right)$$

The Gauss and Codazzi equations

We do not compute in general situation.

Theorem (Theorem 5.3)

The integrability condition of G-W formula is equivalent to the following three equalities:

linear in h & its derivative

$$h_{11,2} - h_{21,1} = \sum_j \left(\Gamma^j_{21} h_{1j} - \Gamma^j_{11} h_{2j} \right)$$

$$h_{12,2} - h_{22,1} = \sum_j \left(\Gamma^j_{22} h_{1j} - \Gamma^j_{12} h_{2j} \right)$$

Codazzi eq.

$$K_{ds^2} = \frac{1}{g}(h_{11}h_{22} - h_{12}h_{21}) (= K)$$

Gauss eq.

Theorema Egregium

(Gaussian curvature) of ds^2

The Gauss and Codazzi equations

Theorem (Theorem 5.3, continued)

Here, $g := \det(g_{ij}) = g_{11}g_{22} - g_{12}g_{21}$, *and*

$$\begin{aligned} K_{ds^2} &:= \frac{1}{g} R_{12}, \\ R_{jk} &:= \frac{1}{2}(g_{1k,2j} - g_{1j,2k} + g_{2j,1k} - g_{2k,1j}) \\ &\quad - \sum_{i,s} g_{is}(\Gamma^s_{ks}\Gamma^i_{1j} - \Gamma^s_{k1}\Gamma^i_{2j}) \\ &\quad + 2\sum_{l,s} g_{kl}(\Gamma^l_{s2}\Gamma^s_{1j} - \Gamma^l_{1s}\Gamma^s_{2j}). \end{aligned}$$

Formulas

- $(g^{ij}) = (g_{ij})^{-1}$

$$\sum_l g^{il} g_{lj} = \delta^i_j, \qquad g^{il}_{,k} = -\sum_{\alpha,\beta} g^{\alpha i} g^{\beta l} g_{\alpha\beta,k}$$

- $\Gamma^k_{ij} = \dfrac{1}{2} \sum_l g^{kl} (g_{lj,i} + g_{il,j} - g_{ij,l}) = \Gamma^k_{ji}$

$$g_{ij,k} = \sum_l \bigl(g_{il}\Gamma^l_{jk} + g_{lj}\Gamma^l_{ik}\bigr), \quad \sum_i \Gamma^i_{ji} = \frac{1}{2g} g_{,j} \quad \bigl(g = \det(g_{ij})\bigr)$$

- $A^i_j = \sum_l g^{il} h_{lj}$

Proof of Theorem 5.3

$$\begin{pmatrix} I_1^1 & I_2^1 & I_3^1 \\ I_1^2 & I_2^2 & I_3^2 \\ I_1^3 & I_2^3 & I_3^3 \end{pmatrix} := \Omega_{1,2} - \Omega_{2,1} - \Omega_1\Omega_2 + \Omega_2\Omega_1$$

$$\Omega_j = \begin{pmatrix} \Gamma_{j1}^1 & \Gamma_{j2}^1 & -A_j^1 \\ \Gamma_{j1}^2 & \Gamma_{j2}^2 & -A_j^2 \\ h_{j1} & h_{j2} & 0 \end{pmatrix} \qquad (j = 1, 2)$$

Exercise 5-1

Problem (Ex. 5-1)

Assume $L = N = 0$, *that is,* $II = 2M\,du\,dv = 2h_{12}\,du^1\,du^2$, *Prove that, if the Gaussian curvature* K *is negative constant,* -1

$$E_v = G_u = 0, \qquad \text{that is,} \qquad g_{11,2} = g_{22,1} = 0.$$

Codazzi eq

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = \sum g_{ij}\,du^i du^j$$

$$\mathrm{II} = 2M\,du\,dv$$

$$K = \frac{\det \hat{\mathrm{II}}}{\det \hat{\mathrm{I}}} = -1$$

Hint $\dfrac{\det \hat{\mathrm{II}}}{\det \hat{\mathrm{I}}} = -1$, $\det \hat{\mathrm{II}} = -M^2$ $\Rightarrow$ $\boxed{4M^2 = EG - F^2}$

- Integrability conditions

$\star$ $\begin{cases} P_{uuv} \cdot \nu & ? \\ P_{uvu} \cdot \nu & ? \end{cases}$

$\begin{cases} P_{vuv} \cdot \nu & ? \\ P_{vvu} \cdot \nu & ? \end{cases}$

$$ds^2 = e^{2\sigma}(du^2 + dv^2)$$

$$\mathrm{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2$$

Problem (Ex. 5-2)

Assume $F = 0$ and $E = G = e^{2\sigma}$, where σ is a function in (u, v). Let $z = u + iv$ $(i = \sqrt{-1})$ and define a complex-valued function q in z by

$$q(z) := \frac{L(u,v) - N(u,v)}{2} - iM(u,v).$$

Prove that the Codazzi equations are equivalent to

$$\frac{\partial q}{\partial \bar{z}} = e^{2\sigma}\frac{\partial H}{\partial z},$$

when $H = \text{cnst}$

$\Rightarrow q_{\bar{z}} = 0$

$z = u + iv$

$\Rightarrow q$: holo in z

where H is the mean curvature, and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

- Gauss & Codazzi for orthonormal frame.

$$\Omega = \begin{pmatrix} 0 & -\sigma_v & -e^{-\sigma}L \\ \sigma_v & 0 & -e^{-\sigma}M \\ e^{-\sigma}L & e^{-\sigma}M & 0 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 0 & -\sigma_u & -e^{-\sigma}M \\ \sigma_u & 0 & -e^{-\sigma}N \\ e^{-\sigma}M & e^{-\sigma}N & 0 \end{pmatrix}$$

skew symm.

$$\underline{\Omega_v - \Lambda_u - \Omega\Lambda + \Lambda\Omega} = 0$$

$$H = \frac{L+N}{2} \cdot e^{-2\sigma} \quad : \text{mean curvature}$$

$$q = \frac{L-N}{2} - iM \qquad z = u + iv$$