Advanced Topics in Geometry A1 (MTH.B405)

The fundamental theorem for surfaces

Kotaro Yamada
kotaro@math.sci.isct.ac.jp
http://www.official.kotaroy.com/class/2025/geom-a1

Institute of Science Tokyo

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$p = p(u^1, u^2)$: a parametrized surface

$\nu = \nu(u^1, u^2)$: the unit normal vector field

$\mathcal{F} = (p_{,1}, p_{,2}, \nu)$: the Gauss Frame

3×3 matrix valued

The Gauss-Weingarten formula:

$$\underline{\underline{\frac{\partial \mathcal{F}}{\partial u^j}}} = \underline{\underline{\mathcal{F}\Omega_j}} \qquad (j = 1, 2)$$

<u>integrability conditions</u>

$$\Omega_j = \begin{pmatrix} \Gamma^1_{1j} & \Gamma^1_{2j} & -A^1_j \\ \Gamma^2_{1j} & \Gamma^2_{2j} & -A^2_j \\ h_{1j} & h_{2j} & 0 \end{pmatrix}$$

The Gauss and Codazzi equations

Codazzi equations

- $$h_{11,2} - h_{21,1} = \sum_j \left(\Gamma^j_{21} h_{1j} - \Gamma^j_{11} h_{2j}\right)$$
- $$h_{12,2} - h_{22,1} = \sum_j \left(\Gamma^j_{22} h_{1j} - \Gamma^j_{12} h_{2j}\right)$$

- $$K_{ds^2} = \frac{1}{g}(h_{11}h_{22} - h_{12}h_{21}) \left(= K\right)$$ Gauss eq

Exercise 5-1

$$\mathrm{II} = \cancel{L\,du^2} + 2M\,du\,dv + \cancel{N\,dv^2}$$

$$2h_{12}\,du^1\,du^2$$

Problem (Ex. 5-1)

Assume $L = N = 0$, *that is,* $II = 2M\,du\,dv = 2h_{12}\,du^1\,du^2$, *Prove that, if the Gaussian curvature* K *is negative constant,*

$$E_v = G_u = 0, \qquad \text{that is,} \qquad g_{11,2} = g_{22,1} = 0.$$

$$K = \frac{\det \hat{\mathrm{II}}}{\det \hat{\mathrm{I}}} = \frac{-M^2}{EG - F^2}$$

$$\hat{\mathrm{II}} = \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}$$

$$\hat{\mathrm{I}} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = (g_{ij})$$

$$\hat{\mathrm{I}}^{-1} = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix} = (g^{ij})$$

Codazzi equations

$$\Leftrightarrow \quad \frac{M_u}{M} = \cancel{M}\left(\Gamma^1_{11} - \Gamma^2_{12}\right)$$

$$\frac{M_v}{M} = \cancel{M}\left(\Gamma^2_{22} - \Gamma^1_{12}\right)$$

$\because$ $M^2 = -K(EG - F^2)$

$$2\frac{M_u}{M} = \frac{(EG - F^2)_u}{EG - F^2}$$

$$2\frac{M_v}{M} = \text{''} \quad v$$

$$\Leftrightarrow \quad \left.\begin{array}{l} F E_v = E G_u \\ F G_u = G E_v \end{array}\right)$$

does not depend on value of K.

$$\Leftrightarrow \quad \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} E_v \\ G_u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

non-singular matrix

$$\Leftrightarrow \quad E_v = G_u = 0$$

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

$$\mathrm{II} = 2M\,du\,dv \qquad (p_u \cdot p_u) \qquad = (p_v \cdot p_v)$$

- $E_v = 0$, $E = E(u) > 0$ $\quad$ $G_u = 0$, $G = G(v) > 0$
- For simplicity assume $K = -1$

$$EG - F^2 = M^2$$

- $\int \sqrt{E}\,du = x \qquad \int \sqrt{G}\,dv = y$ $\qquad \begin{cases} dx = \sqrt{E}\,du \\ dx^2 = E\,du^2 \end{cases}$

$$\Rightarrow ds^2 = 1\,dx^2 + 2\tilde{F}\,dx\,dy + 1\,dy^2 \Rightarrow 2Q$$

$\tilde{F} = \cos\theta$, $\tilde{M} = \sin\theta$

$$\mathrm{II} = 2\tilde{M}\,dx\,dy \qquad 1 - \tilde{F}^2 = \tilde{M}^2$$

Asymptotic ⟶ Chebyshev net

Q and A

Q: I could see in problem 5-1 that E and G are functions of u and v, respectively, but the geometrical meaning is not clear, even with the assumptions of the problem. The assumption of problem 5-1 seems to be a very strong, but not much can be said about its geometrical properties, can it? Can you say much about the geometrical properties?

yes

I'll try to figure out "geometric" properties.

2Q

$ds^2 = e^{2\sigma}(du^2 + dv^2)$.

$\mathrm{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2$

Problem (Ex. 5-2)

Assume $F = 0$ *and* $E = G = e^{2\sigma}$, *where* σ *is a function in* (u, v). *Let* $z = u + iv$ $(i = \sqrt{-1})$ *and define a complex-valued function* q *in* z *by*

Hopf differential

$$✓ \quad q(z) := \frac{L(u,v) - N(u,v)}{2} - iM(u,v).$$

Prove that the Codazzi equations are equivalent to

$$\frac{\partial q}{\partial \bar{z}} = e^{2\sigma} \frac{\partial H}{\partial z},$$

complex numbers

where H *is the mean curvature, and*

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Q: I could see that the calculations would show the conclusion, but where did you come up with this $q(z)$ or whatever it is? Is it a function with a different computational or geometric meaning behind it?

- Why do we use complex parameter $z = u + iv$?

$$ds^2 = \underline{e^{2\sigma}(du^2 + dv^2)} = e^{2\varphi}(d\xi^2 + d\eta^2) \Rightarrow \begin{cases} \xi_u = \pm\eta_v \\ \xi_v = \mp\eta_u \end{cases}$$

"orientation preserving"

$$\det\begin{pmatrix} \xi_u & \xi_v \\ \eta_u & \eta_v \end{pmatrix} > 0 \Rightarrow \boxed{\begin{cases} \xi_u = \eta_v \\ \xi_v = -\eta_u \end{cases}}$$

Cauchy–Riemann

$\Leftrightarrow$ $u + iv \mapsto \xi + i\eta$: holomorphic

$$z = u + iv \qquad \left.\begin{aligned} dz &= du + i\,dv \\ d\bar{z} &= du - i\,dv \end{aligned}\right) \quad \begin{aligned} du &= \frac{1}{2}(dz + d\bar{z}) \\ dv &= \frac{-i}{2}(dz - d\bar{z}) \end{aligned}$$

$$\mathrm{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2$$

$$= \frac{1}{4}\Big(L(dz + d\bar{z})^2 - 2iM(dz + d\bar{z})(dz - d\bar{z}) - N(dz - d\bar{z})^2 \Big)$$

$$= \frac{1}{4}\Big(\underbrace{(L - N - 2iM)}_{2q}\,dz^2 + (L + N)\,dz\,d\bar{z} + (L - N + 2iM)\,d\bar{z}^2 \Big)$$

(2.0) part

(1.1) part

(0.2) part

$$\frac{\partial q}{\partial \bar{z}} = e^{2\sigma}\frac{\partial H}{\partial z},$$

Codazzi equation

C-R. operator

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

<u>Fact</u> $\frac{\partial q}{\partial \bar{z}} = 0 \iff q$ satisfies the Cauchy Rieman

$\iff q$: holomorphic.

<u>Cor</u> If H : const. $\Rightarrow \frac{\partial H}{\partial u} = \frac{\partial H}{\partial v} = 0$

"Codazzi eq" "are" "easy" $\Rightarrow \frac{\partial H}{\partial z} = 0 \Rightarrow q$: <u>holo</u>