Advanced Topics in Geometry A1 (MTH.B405)

The fundamental theorem for surfaces

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The Gauss-Weingarten formulas

$$
\begin{aligned}
p &= p(u^1, u^2) : \text{a parametrized surface} \\
\nu &= \nu(u^1, u^2) : \text{the unit normal vector field} \\
\mathcal{F} &= (p_{,1}, p_{,2}, \nu) : \text{the Gauss Frame}
\end{aligned}
$$

The Gauss-Weingarten formula:

$$
\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j \qquad (j = 1, 2)
$$

The Gauss and Codazzi equations

$$h_{11,2} - h_{21,1} = \sum_j \left(\Gamma^j_{21} h_{1j} - \Gamma^j_{11} h_{2j}\right)$$

$$h_{12,2} - h_{22,1} = \sum_j \left(\Gamma^j_{22} h_{1j} - \Gamma^j_{12} h_{2j}\right)$$

$$K_{ds^2} = \frac{1}{g}(h_{11}h_{22} - h_{12}h_{21}) \big(= K\big)$$

Exercise 5-1

Problem (Ex. 5-1)

Assume $L = N = 0$*, that is,* $II = 2M\,du\,dv = 2h_{12}\,du^1\,du^2$*, Prove that, if the Gaussian curvature* K *is negative constant,*

$$E_v = G_u = 0, \qquad \textit{that is,} \qquad g_{11,2} = g_{22,1} = 0.$$

Q and A

Q: I could see in problem 5-1 that E and G are functions of u and v, respectively, but the geometrical meaning is not clear, even with the assumptions of the problem. The assumption of problem 5-1 seems to be a very strong, but not much can be said about its geometrical properties, can it? Can you say much about the geometrical properties?

Exercise 5-2

Problem (Ex. 5-2)

Assume $F = 0$ *and* $E = G = e^{2\sigma}$*, where* σ *is a function in* (u, v)*. Let* $z = u + iv$ $(i = \sqrt{-1})$ *and define a complex-valued function* q *in* z *by*

$$q(z) := \frac{L(u, v) - N(u, v)}{2} - iM(u, v).$$

Prove that the Codazzi equations are equivalent to

$$\frac{\partial q}{\partial \bar{z}} = e^{2\sigma} \frac{\partial H}{\partial z},$$

where H *is the mean curvature, and*

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Q and A

Q: I could see that the calculations would show the conclusion, but where did you come up with this $q(z)$ or whatever it is? Is it a function with a different computational or geometric meaning behind it?