Advanced Topics in Geometry A1 (MTH.B405)

The fundamental theorem for surfaces

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The fundamental theorem for surfaces

Given data: six functions defined on $U \subset \mathbb{R}^2$.

$$\widehat{I} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad \widehat{II} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},$$

Assumption:

$$g_{11} > 0, \qquad g_{22} > 0, \quad \text{and} \qquad g_{11}g_{22} - g_{12}g_{21} > 0$$

Set up:

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{2} g^{kl}(g_{lj,k} + g_{il,j} - g_{jl,k}), \qquad A_j^i = \sum_{l=1}^{2} g_{jl} h_{il}$$

The statement

Theorem (Theorem 6.1)

Assume U is simply connected, and (g_{ij}) and (h_{ij}) satisfy the Gauss equation and Codazzi equations. Then there exists a regular surface $p\colon U \to \mathbb{R}^3$ such that

- *the first fundamental form of p is $ds^2 = \sum_{i,j} g_{ij} du^i\, du^j$,*
- *the second fundamental form of p with respect to the unit normal vector field $\nu := (p_{,1} \times p_{,2})/|p_{,1} \times p_{,2}|$ coincides with $II = \sum_{i,j} h_{ij} du^i\, du^j$.*

Moreover, such a surface p is unique up to a transformation

$$p \mapsto Rp + \boldsymbol{a}, \qquad R \in \mathrm{SO}(3),\ \boldsymbol{a} \in \mathbb{R}^3.$$

Uniqueness

Existence

Exercise 6-1

Problem (Ex. 6-1)

Let $\theta\colon U \to \mathbb{R}$ be a C^∞-function defined on a simply connected domain U of the uv-plane $\mathbb{R}^2$. Assuming θ satisfies $\theta_{uv} = \sin\theta$, prove that there exists a surface $p\colon U \to \mathbb{R}^3$ whose first and second fundamental forms are

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2, \qquad II = 2\sin\theta\, du\, dv.$$

Exercise 6-2

Problem (Ex. 6-2)

Let $\sigma\colon U \to \mathbb{R}$ be a C^∞-function defined on a simply connected domain U of the uv-plane $\mathbb{R}^2$. Assuming σ satisfies $\Delta\sigma = -\frac{1}{2}\sinh\sigma$, prove that there exists a surface $p\colon U \to \mathbb{R}^3$ with

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad II = \frac{1}{2}\big((e^{2\sigma}+1)du^2 + (e^{2\sigma}-1)dv^2\big).$$