3 Integrability Conditions

Let $U \subset \mathbb{R}^m$ be a domain of $(\mathbb{R}^m; u^1, \dots, u^m)$ and consider an m-tuple of $n \times n$-matrix valued C^∞-maps

$$\Omega_j\colon \mathbb{R}^m \supset U \longrightarrow \mathrm{M}_n(\mathbb{R}) \qquad (j = 1, \dots, m). \tag{3.1}$$

In this section, we consider an initial value problem of a system of linear partial differential equations

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0, \tag{3.2}$$

where $\mathrm{P}_0 = (u_0^1, \dots, u_0^m) \in U$ is a fixed point, X is an $n \times n$-matrix valued unknown, and $X_0 \in \mathrm{M}_n(\mathbb{R})$.

Proposition 3.1. *If a C^∞-map $X\colon U \to \mathrm{M}_n(\mathbb{R})$ defined on a domain $U \subset \mathbb{R}^m$ satisfies* (3.2) *with $X_0 \in \mathrm{GL}(n, \mathbb{R})$, then $X(\mathrm{P}) \in \mathrm{GL}(n, \mathbb{R})$ for all $\mathrm{P} \in U$. In addition, if Ω_j $(j = 1, \dots, m)$ are skew-symmetric and $X_0 \in \mathrm{SO}(n)$, then $X(\mathrm{P}) \in \mathrm{SO}(n)$ holds for all $\mathrm{P} \in U$.*

Proof. Since U is connected, there exists a continuous path $\gamma_0\colon [0, 1] \to U$ such that $\gamma_0(0) = \mathrm{P}_0$ and $\gamma_0(1) = \mathrm{P}$. By Whitney's approximation theorem (cf. Theorem 6.21 in [Lee13]), there exists a smooth path $\gamma\colon [0, 1] \to U$ joining P_0 and P approximating γ_0. Since $\hat{X} := X \circ \gamma$ satisfies (3.4) with $\hat{X}(0) = X_0$, Proposition 2.8 yields that $\det \hat{X}(1) \neq 0$ whenever $\det X_0 \neq 0$. Moreover, if Ω_j's are skew-symmetric, so is $\Omega_\gamma(t)$ in (3.4). Thus, by Proposition 2.10, we obtain the latter half of the proposition. □

Proposition 3.2. *If a matrix-valued C^∞ function $X\colon U \to \mathrm{GL}(n, \mathbb{R})$ satisfies* (3.2)*, it holds that*

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j\Omega_k - \Omega_k\Omega_j \tag{3.3}$$

for each (j, k) with $1 \leqq j < k \leqq m$.

Proof. Differentiating (3.2) by u^k, we have

$$\frac{\partial^2 X}{\partial u^k \partial u^j} = \frac{\partial X}{\partial u^k}\Omega_j + X\frac{\partial \Omega_j}{\partial u^k} = X\left(\frac{\partial \Omega_j}{\partial u^k} + \Omega_k\Omega_j\right).$$

On the other hand, switching the roles of j and k, we get

$$\frac{\partial^2 X}{\partial u^j \partial u^k} = X\left(\frac{\partial \Omega_k}{\partial u^j} + \Omega_j\Omega_k\right).$$

Since X is of class C^∞, the left-hand sides of these equalities coincide, and so are the right-hand sides. Since $X \in \mathrm{GL}(n, \mathbb{R})$, the conclusion follows. □

The equality (3.3) is called the *integrability condition* or *compatibility condition* of (3.2).

The chain rule yields the following:

Lemma 3.3. *Let $X\colon U \to \mathrm{M}_n(\mathbb{R})$ be a C^∞-map satisfying* (3.2)*. Then for each smooth path $\gamma\colon I \to U$ defined on an interval $I \subset \mathbb{R}$, $\hat{X} := X \circ \gamma : I \to \mathrm{M}_n(\mathbb{R})$ satisfies the ordinary differential equation*

$$\frac{d\hat{X}}{dt}(t) = \hat{X}(t)\Omega_\gamma(t) \qquad \left(\Omega_\gamma(t) := \sum_{j=1}^m \Omega_j \circ \gamma(t)\frac{du^j}{dt}(t)\right) \tag{3.4}$$

on I, where $\gamma(t) = \bigl(u^1(t), \dots, u^m(t)\bigr)$.

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Lemma 3.4. *Let $\Omega_j\colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ be C^∞-maps defined on a domain $U \subset \mathbb{R}^m$ which satisfy* (3.3). *Then for each smooth map*

$$\sigma\colon D \ni (t, w) \longmapsto \sigma(t, w) = (u^1(t, w), \dots, u^m(t, w)) \in U$$

defined on a domain $D \subset \mathbb{R}^2$, it holds that

$$\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} - TW + WT = 0, \tag{3.5}$$

where

$$T := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial t}, \quad W := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial w} \quad (\widetilde{\Omega}_j := \Omega_j \circ \sigma). \tag{3.6}$$

Proof. By the chain rule, we have

$$\begin{aligned}
\frac{\partial T}{\partial w} &= \sum_{j,k=1}^m \frac{\partial \Omega_j}{\partial u^k} \frac{\partial u^k}{\partial w} \frac{\partial u^j}{\partial t} + \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial^2 u^j}{\partial w \partial t}, \\
\frac{\partial W}{\partial t} &= \sum_{j,k=1}^m \frac{\partial \Omega_j}{\partial u^k} \frac{\partial u^k}{\partial t} \frac{\partial u^j}{\partial w} + \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial^2 u^j}{\partial t \partial w} \\
&= \sum_{j,k=1}^m \frac{\partial \Omega_k}{\partial u^j} \frac{\partial u^j}{\partial t} \frac{\partial u^k}{\partial w} + \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial^2 u^j}{\partial t \partial w}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} &= \sum_{j,k=1}^m \left(\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} \right) \frac{\partial u^k}{\partial w} \frac{\partial u^j}{\partial t} \\
&= \sum_{j,k=1}^m \left(\widetilde{\Omega}_j \widetilde{\Omega}_k - \widetilde{\Omega}_k \widetilde{\Omega}_j \right) \frac{\partial u^k}{\partial w} \frac{\partial u^j}{\partial t} \\
&= \left(\sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial t} \right) \left(\sum_{k=1}^m \widetilde{\Omega}_k \frac{\partial u^k}{\partial w} \right) - \left(\sum_{k=1}^m \widetilde{\Omega}_k \frac{\partial u^k}{\partial w} \right) \left(\sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial t} \right) \\
&= TW - WT.
\end{aligned}$$

Thus (3.5) holds.

Integrability of linear systems. The main theorem in this section is the following theorem:

Theorem 3.5. *Let $\Omega_j\colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ be C^∞-functions defined on a* <u>*simply connected*</u> *domain $U \subset \mathbb{R}^m$ satisfying* (3.3). *Then for each $\mathrm{P}_0 \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique $n \times n$-matrix valued function $X\colon U \to \mathrm{M}_n(\mathbb{R})$ satisfying* (3.2). *Moreover,*

- *if $X_0 \in \mathrm{GL}(n, \mathbb{R})$, $X(\mathrm{P}) \in \mathrm{GL}(n, \mathbb{R})$ holds on U,*
- *if $X_0 \in \mathrm{SO}(n)$ and Ω_j $(j = 1, \dots, m)$ are skew-symmetric matrices, $X \in \mathrm{SO}(n)$ holds on U.*

Proof. The latter half is a direct conclusion of Proposition 3.1. We show the existence of X: Take a smooth path $\gamma\colon [0, 1] \to U$ joining P_0 and P. Then by Theorem 2.15, there exists a unique C^∞-map $\hat{X}\colon [0, 1] \to \mathrm{M}_n(\mathbb{R})$ satisfying (3.4) with initial condition $\hat{X}(0) = X_0$.

We shall show that the value $\hat{X}(1)$ does not depend on choice of paths joining P_0 and P. To show this, choose another smooth path $\tilde{\gamma}$ joining P_0 and P. Since U is simply connected, there exists a homotopy between γ and $\tilde{\gamma}$, that is, there exists a continuous map $\sigma_0\colon [0,1]\times[0,1]\ni(t,w)\mapsto\sigma_0(t,w)\in U$ satisfying

$$\begin{aligned}\sigma_0(t,0)&=\gamma(t), & \sigma_0(t,1)&=\tilde{\gamma}(t),\\ \sigma_0(0,w)&=\mathrm{P}_0, & \sigma_0(1,w)&=\mathrm{P}.\end{aligned}\tag{3.7}$$

Then, by Whitney's approximation theorem (Theorem 6.21 in [Lee13]) again, there exists a smooth map $\sigma\colon[0,1]\times[0,1]\to U$ satisfying the same boundary conditions as (3.7):

$$\begin{aligned}\sigma(t,0)&=\gamma(t), & \sigma(t,1)&=\tilde{\gamma}(t),\\ \sigma(0,w)&=\mathrm{P}_0, & \sigma(1,w)&=\mathrm{P}.\end{aligned}\tag{3.8}$$

We set T and W as in (3.6). For each fixed $w\in[0,1]$, there exists $X_w\colon[0,1]\to\mathrm{M}_n(\mathbb{R})$ such that

$$\frac{dX_w}{dt}(t)=X_w(t)T(t,w),\qquad X_w(0)=X_0.$$

Since $T(t,w)$ is smooth in t and w, the map

$$\check{X}\colon[0,1]\times[0,1]\ni(t,w)\mapsto X_w(t)\in\mathrm{M}_n(\mathbb{R})$$

is a smooth map, because of smoothness in parameter α in Theorem 2.15. To show that $\hat{X}(1)=\check{X}(1,0)$ does not depend on choice of paths, it is sufficient to show that

$$\frac{\partial\check{X}}{\partial w}=\check{X}W\tag{3.9}$$

holds on $[0,1]\times[0,1]$. In fact, by (3.8), $W(1,w)=0$ for all $w\in[0,1]$, and then (3.9) implies that $\check{X}(1,w)$ is constant.

We prove (3.9): By definition, it holds that

$$\frac{\partial\check{X}}{\partial t}=\check{X}T,\qquad\check{X}(0,w)=X_0\tag{3.10}$$

for each $w\in[0,1]$. Hence by (3.5),

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial\check{X}}{\partial w}&=\frac{\partial^2\check{X}}{\partial t\partial w}=\frac{\partial^2\check{X}}{\partial w\partial t}=\frac{\partial}{\partial w}(\check{X}T)\\ &=\frac{\partial\check{X}}{\partial w}T+\check{X}\frac{\partial T}{\partial w}=\frac{\partial\check{X}}{\partial w}T+\check{X}\left(\frac{\partial W}{\partial t}+TW-WT\right)\\ &=\frac{\partial\check{X}}{\partial w}T+\check{X}\frac{\partial W}{\partial t}+\frac{\partial\check{X}}{\partial t}W-\check{X}WT\\ &=\frac{\partial}{\partial t}(\check{X}W)+\left(\frac{\partial\check{X}}{\partial w}-\check{X}W\right)T.\end{aligned}$$

So, the function $Y_w(t):=\partial\check{X}/\partial w-\check{X}W$ satisfies the ordinary differential equation

$$\frac{dY_w}{dt}(t)=Y_w(t)T(t,w),\quad Y_w(0)=O$$

for each $w\in[0,1]$. Thus, by the uniqueness of the solution, $Y_w(t)=O$ holds on $[0,1]\times[0,1]$. Hence we have (3.9).

Thus, $\hat{X}(1)$ depends only on the end point P of the path. Hence we can set $X(\mathrm{P}):=\hat{X}(1)$ for each $\mathrm{P}\in U$, and obtain a map $X\colon U\to\mathrm{M}_n(\mathbb{R})$. Finally we show that X is the desired solution. The initial condition $X(\mathrm{P}_0)=X_0$ is obviously satisfied. On the other hand, if we set

$$Z(\delta):=X(u^1,\dots,u^j+\delta,\dots,u^m),$$

$Z(\delta)$ satisfies the equation (3.4) for the path $\gamma(\delta) := (u^1, \dots, u^j + \delta, \dots, u^m)$ with $Z(0) = X(\mathrm{P})$. Since $\Omega_\gamma = \Omega_j$,

$$\frac{\partial X}{\partial u^j}(\mathrm{P}) = \left.\frac{dZ}{d\delta}\right|_{\delta=0} = Z(0)\Omega_j(\mathrm{P}) = X(\mathrm{P})\Omega_j(\mathrm{P})$$

which completes the proof. □

Application: Poincaré's lemma.

Theorem 3.6 (Poincaré's lemma)**.** *If a differential* 1*-form*

$$\omega = \sum_{j=1}^{m} \alpha_j(u^1, \dots, u^m)\, du^j$$

defined on a simply connected domain $U \subset \mathbb{R}^m$ is closed, that is, $d\omega = 0$ holds, then there exists a C^∞-function f on U such that $df = \omega$. Such a function f is unique up to additive constants.

Proof. Since

$$d\omega = \sum_{i<j} \left(\frac{\partial \alpha_j}{\partial u^i} - \frac{\partial \alpha_i}{\partial u^j} \right) du^i \wedge du^j,$$

the assumption is equivalent to

$$\frac{\partial \alpha_j}{\partial u^i} - \frac{\partial \alpha_i}{\partial u^j} = 0 \qquad (1 \leqq i < j \leqq m). \tag{3.11}$$

Consider a system of linear partial differential equations with unknown ξ, a 1×1-matrix valued function (i.e. a real-valued function), as

$$\frac{\partial \xi}{\partial u^j} = \xi \alpha_j \quad (j = 1, \dots, m), \qquad \xi(u_0^1, \dots, u_0^m) = 1. \tag{3.12}$$

Then it satisfies (3.3) because of (3.11). Hence by Theorem 3.5, there exists a smooth function $\xi(u^1, \dots, u^m)$ satisfying (3.12). In particular, Proposition 2.8 yields $\xi = \det \xi$ never vanishes. Hence $\xi(u_0^1, \dots, u_0^m) = 1 > 0$ means that $\xi > 0$ holds on U. Letting $f := \log \xi$, we have the function f satisfying $df = \omega$.

Next, we show the uniqueness: if two functions f and g satisfy $df = dg = \omega$, it holds that $d(f - g) = 0$. Hence by connectivity of U, $f - g$ must be constant. □

Application: Conjugation of Harmonic functions. In this paragraph, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. It is well-known that a smooth function

$$f\colon U \ni u + \mathrm{i}\, v \longmapsto \xi(u, v) + \mathrm{i}\, \eta(u, v) \in \mathbb{C} \qquad (\mathrm{i} = \sqrt{-1}) \tag{3.13}$$

defined on a domain $U \subset \mathbb{C}$ is *holomorphic* if and only if it satisfies the following relation, called the *Cauchy-Riemann equations*:

$$\frac{\partial \xi}{\partial u} = \frac{\partial \eta}{\partial v}, \qquad \frac{\partial \xi}{\partial v} = -\frac{\partial \eta}{\partial u}. \tag{3.14}$$

Definition 3.7. A function $f\colon U \to \mathbb{R}$ defined on a domain $U \subset \mathbb{R}^2$ is said to be *harmonic* if it satisfies

$$\Delta f = f_{uu} + f_{vv} = 0.$$

The operator Δ is called the *Laplacian*.

Proposition 3.8. *If function f in* (3.13) *is holomorphic, $\xi(u,v)$ and $\eta(u,v)$ are harmonic functions.*

Proof. By (3.14), we have

$$\xi_{uu} = (\xi_u)_u = (\eta_v)_u = \eta_{vu} = \eta_{uv} = (\eta_u)_v = (-\xi_v)_v = -\xi_{vv}.$$

Hence $\Delta\xi = 0$. Similarly,

$$\eta_{uu} = (-\xi_v)_u = -\xi_{vu} = -\xi_{uv} = -(\xi_u)_v = -(\eta_v)_v = -\eta_{vv}.$$

Thus $\Delta\eta = 0$. □

Theorem 3.9. *Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u,v)$ a C^∞-function harmonic on U*[8] *Then there exists a C^∞ harmonic function η on U such that $\xi(u,v) + \mathrm{i}\,\eta(u,v)$ is holomorphic on U.*

Proof. Let $\alpha := -\xi_v\,du + \xi_u\,dv$. Then by the assumption,

$$d\alpha = (\xi_{vv} + \xi_{uu})\,du \wedge dv = 0$$

holds, that is, α is a closed 1-form. Hence by simple connectivity of U and the Poincaré's lemma (Theorem 1.9), there exists a function η such that $d\eta = \eta_u\,du + \eta_v\,dv = \alpha$. Such a function η satisfies (3.14) for given ξ. Hence $\xi + \mathrm{i}\,\eta$ is holomorphic in $u + \mathrm{i}\,v$. □

Example 3.10. A function $\xi(u,v) = e^u \cos v$ is harmonic. Set

$$\alpha := -\xi_v\,du + \xi_u\,dv = e^u \sin v\,du + e^u \cos v\,dv.$$

Then $\eta(u,v) = e^u \sin v$ satisfies $d\eta = \alpha$. Hence

$$\xi + \mathrm{i}\,\eta = e^u(\cos v + \mathrm{i} \sin v) = e^{u+\mathrm{i}\,v}$$

is holomorphic in $u + \mathrm{i}\,v$.

Definition 3.11. The harmonic function η in Theorem 3.9 is called the *conjugate* harmonic function of ξ.

[8] The theorem holds under the assumption of C^2-differentiability.

Exercises

3-1 Let

$$\xi_1(u,v) := \frac{u}{u^2+v^2}, \qquad \xi_2(u,v) := \log\sqrt{u^2+v^2}$$

be functions defined on <u>non-simply connected</u> domain $U := \mathbb{R}^2 \setminus \{(0,0)\}$.

(1) Show that both ξ_1 and ξ_2 are harmonic on U.

(2) Verify that there exists a conjugate harmonic function η_1 of ξ_1 on U.

(3) Prove that there exists no conjugate harmonic function η_2 of ξ_2 on U.

3-2 Consider a linear system of partial differential equationcs for 3×3-matrix valued unknown X on a domain $U \subset \mathbb{R}^2$ as

$$\frac{\partial X}{\partial u} = X\Omega, \quad \frac{\partial X}{\partial v} = X\Lambda, \qquad \left(\Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix}\right),$$

where (u,v) are the canonical coordinate system of $\mathbb{R}^2$, and α, β and h_j^i $(i,j = 1,2)$ are smooth functions defined on U. Write down the integrability conditions in terms of α, β and h_j^i.