# 5 The Gauss and Codazzi equations

### *5.1 Gauss and Codazzi equations*

The Gauss-Weingarten formulas (Theorem 4.2) can be considered as a system of partial differential equations with unknown $\mathcal{F}$, whose coefficient matrices are $\Omega_1$ and $\Omega_2$.

*Remark* 5.1. The coefficient matrices $\Omega_1$ and $\Omega_2$ in the Gauss-Weingarten formula are expressed in terms of the coefficients of the first and second fundamental forms. In fact, explicit formula for components of $\Omega_j$ in terms of $(g_{ij})$ and $(h_{ij})$ are found in (4.15) and (4.18).

The following proposition is a direct conclusion of Proposition 3.2 and Theorem 4.2:

**Proposition 5.2.** *Let $p\colon U \to \mathbb{R}^3$ be a parametrized surface defined on a domain $U$ of $u^1u^2$-plane, and let $(g_{ij})$ and $(h_{ij})$ be the coefficients of the first and second fundamental forms. Then the matrices $\Omega_1$ and $\Omega_2$ in* (4.17) *satisfy*

$$\frac{\partial \Omega_1}{\partial u^2} - \frac{\partial \Omega_2}{\partial u^1} - \Omega_1\Omega_2 + \Omega_2\Omega_1 = O \tag{5.1}$$

In this section, we show that nine equalities (5.1) are reduced to three equalities, as follows:

**Theorem 5.3** (Gauss and Codazzi equations)**.** *The integrability condition* (5.1) *is equivalent to the following three equalities:*

$$h_{11,2} - h_{21,1} = \sum_j \left(\Gamma^j_{21}h_{1j} - \Gamma^j_{11}h_{2j}\right) \tag{5.2}$$

$$h_{12,2} - h_{22,1} = \sum_j \left(\Gamma^j_{22}h_{1j} - \Gamma^j_{12}h_{2j}\right) \tag{5.3}$$

$$K_{ds^2} = \frac{1}{g}(h_{11}h_{22} - h_{12}h_{21})\bigl(= K\bigr), \tag{5.4}$$

*where $g := \det(g_{ij}) = g_{11}g_{22} - g_{12}g_{21}$, and*

$$K_{ds^2} := \frac{1}{g}R_{12}, \tag{5.5}$$

$$\begin{aligned} R_{jk} := {}& \frac{1}{2}(g_{1k,2j} - g_{1j,2k} + g_{2j,1k} - g_{2k,1j}) - \sum_{i,s} g_{is}(\Gamma^s_{k2}\Gamma^i_{1j} - \Gamma^s_{k1}\Gamma^i_{2j}) \\ & + 2\sum_{l,s} g_{kl}(\Gamma^l_{s2}\Gamma^s_{1j} - \Gamma^l_{1s}\Gamma^s_{2j}). \end{aligned} \tag{5.6}$$

The equalities (5.2) and (5.3) are called the *Codazzi equations* , and (5.4) is called the *Gauss equation*.

*Remark* 5.4. Let

$$h_{ij;k} := h_{ij,k} - \sum_l \left(\Gamma^l_{ik}h_{lj} - \Gamma^l_{kj}h_{il}\right).$$

Then

$$\nabla II := \sum_{i,j,k} h_{ij;k}\,du^i \otimes du^j \otimes du^k$$

does not depend on the coordinate system, which is called the *covariant derivative* of the second fundamental form. The Codazzi equations is equivalent to $h_{ij;k} = h_{ki;j}$, that is, symmetricity of $\nabla II$.

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*Remark* 5.5. The quantity $K_{ds^2}$ in (5.5) is determined only by the first fundamental form, and one can show that it is invariant under coordinate changes. We call it the (intrinsic) *Gaussian curvature* of $ds^2$. The Gauss equation (5.4) claims that the intrinsic Gaussian curvature is identical to the Gaussian curvature of the surface.

*Proof of Theorem 5.3.* We set

$$\begin{pmatrix} I_1^1 & I_2^1 & I_3^1 \\ I_1^2 & I_2^2 & I_3^2 \\ I_1^3 & I_2^3 & I_3^3 \end{pmatrix} := \Omega_{1,2} - \Omega_{2,1} - \Omega_1\Omega_2 + \Omega_2\Omega_1.$$

Then the integrability condition (5.1) is equivalent to $I_j^i = 0$ $(i, j = 1, 2, 3)$.
*Step 1.* By symmetricity of $h_{ij}$ and $g^{ij}$,

$$\begin{aligned} I_3^3 &= h_{11}A_2^1 + h_{12}A_2^2 - h_{21}A_1^1 - h_{22}A_1^2 = \sum_l (h_{1l}A_2^l - h_{2l}A_1^l) \\ &= \sum_l \left( h_{1l}\sum_s g^{ls}h_{s2} - h_{2l}\sum_s g^{ls}h_{s1} \right) \\ &= \sum_{l,s} g^{ls}h_{1l}h_{s2} - \sum_{l,s} g^{ls}h_{s1}h_{2l} = \sum_{l,s} g^{ls}h_{1l}h_{s2} - \sum_{l,s} g^{sl}h_{l1}h_{2s} = 0. \end{aligned}$$

Thus the condition $I_3^3 = 0$ is satisfied automatically.
*Step 2.* Since

$$I_j^3 = h_{1j,2} - h_{2j,1} - \sum_l (\Gamma_{2j}^l h_{l1} - \Gamma_{1j}^l h_{l2}) \qquad (j = 1, 2),$$

the conditions $I_j^3 = 0$ $(j = 1, 2)$ are equivalent to the Codazzi equations (5.2) and (5.3).
*Step 3.* For $j = 1, 2$

$$\begin{aligned} I_3^j &= -A_{1,2}^j + A_{2,1}^j + \sum_l (\Gamma_{1l}^j A_2^l - \Gamma_{2l}^j A_1^l) \\ &= -\sum_l (g^{jl}h_{1l})_{,2} + \sum_l (g^{jl}h_{l2})_{,1} + \sum_{l,s} g^{ls}(h_{2s}\Gamma_{1l}^j - h_{1s}\Gamma_{2l}^j) \\ &= -\sum_l g^{jl}(h_{1l,2} - h_{l2,1}) - \sum_l (g_{,2}^{jl}h_{1l} - g_{,1}^{jl}h_{l2}) + \sum_{l,s} g^{ls}(h_{2s}\Gamma_{1l}^j - h_{1s}\Gamma_{2l}^j) \\ &= -\sum_l g^{jl}(h_{1l,2} - h_{l2,1}) + \sum_l \sum_{\alpha,\beta} g^{\alpha j}g^{l\beta}(g_{\alpha\beta,2}h_{1l} - g_{\alpha\beta,1}h_{l2}) + \sum_{l,s} g^{ls}(h_{2s}\Gamma_{1l}^j - h_{1s}\Gamma_{2l}^j) \\ &= -\sum_l g^{jl}(h_{1l,2} - h_{l2,1}) + \sum_{l,\alpha,\beta} g^{\alpha j}g^{l\beta}\sum_s \big((g_{\alpha s}\Gamma_{\beta 2}^s + g_{s\beta}\Gamma_{2\beta}^s)h_{1l} - (g_{\alpha s}\Gamma_{\beta 1}^s + g_{s\beta}\Gamma_{1\beta}^s)h_{l2}\big) \\ &\qquad + \sum_{l,s} g^{ls}(h_{2s}\Gamma_{1l}^j - h_{1s}\Gamma_{2l}^j) \\ &= -\sum_l g^{jl}(h_{1l,2} - h_{l2,1}) + \sum_{l,\beta} g^{l\beta}\Gamma_{\beta 2}^j h_{1l} + \sum_{l,\alpha} g^{j\alpha}\Gamma_{\alpha 2}^l h_{1l} - \sum_{l,\beta} g^{l\beta}\Gamma_{\beta 1}^j h_{2l} - \sum_{l,\alpha} g^{j\alpha}\Gamma_{\alpha 1}^l h_{2l} \\ &\qquad + \sum_{l,s} g^{ls}(h_{2s}\Gamma_{1l}^j - h_{1s}\Gamma_{2l}^j) \\ &= -\sum_l g^{jl}(h_{1l,2} - h_{l2,1}) - \sum_s (\Gamma_{l2}^s h_{1s} - \Gamma_{1l}^s h_{2s}) = -\sum_l g^{jl}I_l^3, \end{aligned}$$

that is,

$$\begin{pmatrix} I_3^1 \\ I_3^2 \end{pmatrix} = -\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} \begin{pmatrix} I_1^3 \\ I_2^3 \end{pmatrix}.$$

Here, we used Proposition 4.3 and the relation $\widehat{I}^{-1}_{,k} = -\widehat{I}^{-1}\,\widehat{I}_{,k}\,\widehat{I}^{-1}$, i.e.,

$$g^{ij}_{,k} = -\sum_{\alpha\beta} g^{\alpha i} g^{j\beta} g_{\alpha\beta,k}$$

Hence the conditions $I^j_3 = 0$ $(j = 1, 2)$ are equivalent to $I^3_j = 0$ $(j = 1, 2)$.
<u>*Step 4.*</u> Since

$$I^i_j = \Gamma^i_{1j,2} - \Gamma^i_{2j,1} - \sum_l (\Gamma^i_{1l}\Gamma^l_{2j} - \Gamma^i_{2l}\Gamma^l_{1j}) + A^i_1 h_{j2} - A^i_2 h_{j1},$$

for $i, j = 1, 2$, we have

$$\sum_i g_{ik} I^i_j = (h_{l1}h_{j2} - h_{l2}h_{j1}) = R_{jk} + h_{k1}h_{j2} - h_{k2}h_{j1},$$

where $R_{jk}$ is the quantity given by (5.6). Since the right-most term of the definition of $R_{jk}$ is computed as

$$\sum_{l,s} g_{kl}(\Gamma^s_{1j}\Gamma^l_{s2} - \Gamma^s_{2j}\Gamma^l_{s1}) = \frac{1}{2}\sum_{s,t}\big((g_{k2,s} + g_{sk,2} - g_{2s,k})(g_{tj,1} + g_{1t,j} - g_{1j,t}) \\ - (g_{k1,t} + g_{tk,2} - g_{1k,t})(g_{sj,22} + g_{2s,j} - g_{2j,s})\big),$$

Hence $R_{jk}$ is skew symmetric in $j$ and $k$:

$$R_{12} = -R_{21}, \qquad R_{11} = R_{22} = 0.$$

Therefore $I^i_j = 0$ for $i, j = 1, 2$ is equivalent to the Gauss equation (5.4). □

### *5.2 Integrability conditions for orthonormal frames*

Under the formulation with orthonormal frame as in Proposition 4.8, we can compute the integrability conditions. Since $\Omega$ and $\Lambda$ are skew-symmetric matrices, the conditions consist of three scalar equalities obviously. Such a formulation will be discussed in the lecture on the next quarter.

***Exercises***

Let $p\colon U \to \mathbb{R}^3$ be a regular surface of domain $U \subset \mathbb{R}^2$, and denote by $(u^1, u^2) = (u, v)$ the coordinate system of $U$. And write the first and second fundamental forms as

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = \sum_{i,j} g_{ij}\,du^i\,du^j,$$
$$II = L\,du^2 + 2M\,du\,dv + N\,dv^2 = \sum_{i,j} h_{ij}\,du^i\,du^j,$$

respectively.

**5-1** Assume $L = N = 0$, that is, $II = 2M\,du\,dv = 2h_{12}\,du^1\,du^2$, Prove that, if the Gaussian curvature $K$ is negative constant,

$$E_v = G_u = 0, \qquad \text{that is,} \qquad g_{11,2} = g_{22,1} = 0.$$

**5-2** Assume $F = 0$ and $E = G = e^{2\sigma}$, where $\sigma$ is a function in $(u, v)$. Let $z = u + iv$ ($i = \sqrt{-1}$) and define a complex-valued function $q$ in $z$ by

$$q(z) := \frac{L(u,v) - N(u,v)}{2} - iM(u,v).$$

Prove that the Codazzi equations are equivalent to

$$\frac{\partial q}{\partial \bar{z}} = e^{2\sigma}\frac{\partial H}{\partial z},$$

where $H$ is the mean curvature, and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$