6 The fundamental theorem for surfaces

6.1 The statement

Let U be a domain of u^1u^2-plane and let

$$\widehat{I} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad \widehat{II} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \tag{6.1}$$

be two symmetric matrices whose components are real-valued C^∞-functions on U. In addition, assume

$$g_{11} > 0, \qquad g_{22} > 0, \quad \text{and} \qquad g_{11}g_{22} - g_{12}g_{21} > 0 \tag{6.2}$$

hold on U. In other words, $\widehat{I}$ is a positive-definite matrix at each point on U. Define

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{2} g^{kl}(g_{kj,i} + g_{ik,j} - g_{ij,k}), \qquad A^i_j = \sum_{l=1}^{2} g^{il}h_{lj} \tag{6.3}$$

where $(g^{ij}) = (g_{ij})^{-1}$ is the inverse matrix of (g_{ij}).

Theorem 6.1 (The fundamental theorem for surface theory)**.** *Assume U is simply connected, and (g_{ij}) and (h_{ij}) satisfy the Gauss equation* (5.4) *and the Codazzi equations* (5.2)–(5.3) *in the previous section. Then there exists a regular surface $p\colon U \to \mathbb{R}^3$ such that*

- *the first fundamental form of p is $ds^2 = \sum_{i,j} g_{ij}du^i\,du^j$,*
- *the second fundamental form of p with respect to the unit normal vector field $\nu := (p_{,1} \times p_{,2})/|p_{,1} \times p_{,2}|$ coincides with $II = \sum_{i,j} h_{ij}du^i\,du^j$.*

Moreover, such a surface p is unique up to a transformation

$$p \mapsto Rp + \boldsymbol{a}, \qquad R \in \mathrm{SO}(3),\ \boldsymbol{a} \in \mathbb{R}^3.$$

6.2 Uniqueness

Here we shall prove the uniqueness part of Theorem 6.1. Let p and $\tilde{p}$ be regular surfaces in $\mathbb{R}^3$ defined on a domain U of u^1u^2-plane[10], with unit normal vector fields

$$\nu := \frac{p_{,1} \times p_{,2}}{|p_{,1} \times p_{,2}|} \qquad \text{and} \qquad \tilde{\nu} := \frac{\tilde{p}_{,1} \times \tilde{p}_{,2}}{|\tilde{p}_{,1} \times \tilde{p}_{,2}|},$$

respectively. Then the Gauss frame of p and $\tilde{p}$ are written as

$$\mathcal{F} := (p_{,1}, p_{,2}, \nu), \qquad \text{and} \qquad \widetilde{\mathcal{F}} := (\tilde{p}_{,1}, \tilde{p}_{,2}, \tilde{\nu}),$$

respectively. Assume the coefficients (g_{ij}) and $(h_{ij})$of the first and second fundamental forms are common for p and $\tilde{p}$. Then $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ satisfy the Gauss-Weingarten equations (4.17)

$$\mathcal{F}_{,j} = \mathcal{F}\Omega_j \quad \text{and} \quad \widetilde{\mathcal{F}}_{,j} = \widetilde{\mathcal{F}}\Omega_j, \qquad \text{where} \quad \Omega_j = \begin{pmatrix} \Gamma^1_{1j} & \Gamma^1_{2j} & -A^1_j \\ \Gamma^2_{1j} & \Gamma^2_{2j} & -A^2_j \\ h_{1j} & h_{2j} & 0 \end{pmatrix}. \tag{6.4}$$

Hence, for $i = 1, 2$,

$$\frac{\partial}{\partial u^j}\widetilde{\mathcal{F}}\mathcal{F}^{-1} = \widetilde{\mathcal{F}}_{,j}\mathcal{F}^{-1} + \widetilde{\mathcal{F}}(\mathcal{F}^{-1})_{,j} = \widetilde{\mathcal{F}}_{,j}\mathcal{F}^{-1} - \widetilde{\mathcal{F}}\mathcal{F}^{-1}\mathcal{F}_{,j}\mathcal{F}^{-1} = \widetilde{\mathcal{F}}\Omega_j\mathcal{F}^{-1} - \widetilde{\mathcal{F}}\Omega_j\mathcal{F}^{-1} = O$$

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[10]The uniqueness does not require simple connectedness of U.

hold on U. Since we have assumed that U is a domain, U is (arcwise) connected. This implies that $R := \widetilde{\mathcal{F}}\mathcal{F}^{-1}$ is a constant matrix on U. Moreover, since p and $\tilde{p}$ share their first fundamental forms, it holds that

$$\mathcal{F}^T\mathcal{F} = \begin{pmatrix} p_{,1}\cdot p_{,1} & p_{,1}\cdot p_{,2} & p_{,1}\cdot\nu \\ p_{,2}\cdot p_{,1} & p_{,2}\cdot p_{,2} & p_{,2}\cdot\nu \\ \nu\cdot p_{,1} & \nu\cdot p_{,2} & \nu\cdot\nu \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \widetilde{\mathcal{F}}^T\widetilde{\mathcal{F}} = \mathcal{F}^T R^T R\mathcal{F}.$$

Hence $R^TR = \mathrm{id}$, that is, R is an orthogonal matrix. Moreover,

$$\tilde{\nu} = \frac{\tilde{p}_{,1}\times\tilde{p}_{,2}}{|\tilde{p}_{,1}\times\tilde{p}_{,2}|} = R\nu = R\frac{p_{,1}\times p_{,2}}{|p_{,1}\times p_{,2}|}$$

implies $R(p_{,1}\times p_{,2}) = (Rp_{,1})\times(Rp_{,2})$, hence $\det R = 1$. Summing up, the Gauss frames $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are related as $\widetilde{\mathcal{F}} = R\mathcal{F}$ ($R\in\mathrm{SO}(3)$). By the first and second columns of this relation, it holds that

$$d\tilde{p} = \tilde{p}_{,1}\,du^1 + \tilde{p}_{,2}\,du^2 = Rp_{,1}du^1 + Rp_{,2}\,du^2 = R(dp).$$

Hence, by connectivitity of U again, $\boldsymbol{a} := \tilde{p} - Rp$ is a constant vector. □

6.3 Existence

Next, we show the existence part of Theorem 6.1.

Lemma 6.2. *Let (γ_{ij}) be a positive definite symmetric matrix, that is, γ_{11} and γ_{22} are positive, $\gamma_{11}\gamma_{22}-\gamma_{12}\gamma_{21} > 0$ and $\gamma_{12}=\gamma_{21}$. Then there exists a vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ in $\mathbb{R}^3$ such that*

$$\boldsymbol{v}_i\cdot\boldsymbol{v}_j = \gamma_{ij}, \qquad \boldsymbol{v}_3\cdot\boldsymbol{v}_j = 0, \qquad \boldsymbol{v}_3\cdot\boldsymbol{v}_3 = 1, \qquad \textit{and} \qquad \det(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3) > 0$$

hold for i, $j = 1, 2$.

Proof. Let $\theta\in(0,\pi)$ be an angle satisfying $\cos\theta = g_{12}/\sqrt{g_{11}g_{22}} \in (-1,1)\setminus\{0\}$, and set

$$\boldsymbol{v}_1 := \sqrt{g_{11}}\begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad \boldsymbol{v}_2 := \sqrt{g_{22}}\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix}, \qquad \boldsymbol{v}_3 := \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Then $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are desired vectors. □

Step 1. We fix a point P_0 in U. Then by Lemma 6.2, there exists a matrix $\mathcal{F}_0$ such that

(6.5)
$$\mathcal{F}_0^T\mathcal{F}_0 = \begin{pmatrix} g_{11}(\mathrm{P}_0) & g_{12}(\mathrm{P}_0) & 0 \\ g_{21}(\mathrm{P}_0) & g_{22}(\mathrm{P}_0) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since (g_{ij}) and (h_{ij}) satisfy the Gauss and Codazzi equations, Theorem 5.3 implies that the equation (6.4) for unknown matrix-valued function $\mathcal{F}$. So, by Theorem 3.5, there exists a unique matrix-valued function $\mathcal{F}$ defined on U satisfying

(6.6)
$$\mathcal{F}_{,j} = \mathcal{F}\Omega_j, \qquad \mathcal{F}(\mathrm{P}_0) = \mathcal{F}_0$$

for a matrix $\mathcal{F}_0$ satisfying (6.5). Decompose the solution $\mathcal{F}$ into column vectors as

$$\mathcal{F}(u^1,u^2) = (\boldsymbol{a}_1(u^1,u^2), \boldsymbol{a}_2(u^1,u^2), \boldsymbol{a}_3(u^1,u^2)).$$

Then it hold that

$$\frac{\partial}{\partial u^2}(\boldsymbol{a}_1) = \mathit{\Gamma}^1_{12}\boldsymbol{a}_1 + \mathit{\Gamma}^2_{12}\boldsymbol{a}_2 + h_{12}\boldsymbol{a}_3,$$
$$\frac{\partial}{\partial u^1}(\boldsymbol{a}_2) = \mathit{\Gamma}^1_{21}\boldsymbol{a}_1 + \mathit{\Gamma}^2_{21}\boldsymbol{a}_2 + h_{21}\boldsymbol{a}_3,$$

that is,

$$\omega := \boldsymbol{a}_1\,du^1 + \boldsymbol{a}_2\,du^2$$

is a (vector-valued) closed one-form on the simply connected domain U. Hence by Poincaré's lemma (Theorem 1.9), there exists a map $p\colon U \to \mathbb{R}^3$ with $dp = \omega$, that is,

$$p_{,1} = \boldsymbol{a}_1, \qquad p_{,2} = \boldsymbol{a}_2. \tag{6.7}$$

Step 2. We shall show that p obtained in the previous step is the desired one. Let $\mathcal{F}$ be a solution of (6.6). Then the symmetric matrix-valued function $\mathcal{F}^T\mathcal{F}$ satisfies a system of linear partial differential equations

$$\frac{\partial \mathcal{F}^T\mathcal{F}}{\partial u^j} = \mathit{\Omega}_j^T\mathcal{F}^T\mathcal{F} + \mathcal{F}^T\mathcal{F}\mathit{\Omega}_j, \qquad \mathcal{F}^T\mathcal{F}(\mathrm{P}_0) = \mathcal{F}_0^T\mathcal{F}_0$$

where $\mathcal{F}_0$ is a matrix as in (6.5).

On the other hand, consider the matrix-valued function

$$\mathcal{G} := \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, by (6.3), it holds that

$$\mathcal{G}_{,j} = \mathit{\Omega}_j^T\mathcal{G} + \mathcal{G}\mathit{\Omega}_j \qquad \mathcal{G}(\mathrm{P}_0) = \mathcal{F}_0^T\mathcal{F}_0. \tag{6.8}$$

Hence $\mathcal{F}^T\mathcal{F}$ and $\mathcal{G}$ satisfy the same system of partial differential equations with the same initial conditions. Thus, the uniqueness of the solution infers $\mathcal{F}^T\mathcal{F} = \mathcal{G}$, that is,

$$\begin{pmatrix} \boldsymbol{a}_1\cdot\boldsymbol{a}_1 & \boldsymbol{a}_1\cdot\boldsymbol{a}_2 & \boldsymbol{a}_1\cdot\boldsymbol{a}_3 \\ \boldsymbol{a}_2\cdot\boldsymbol{a}_1 & \boldsymbol{a}_2\cdot\boldsymbol{a}_2 & \boldsymbol{a}_2\cdot\boldsymbol{a}_3 \\ \boldsymbol{a}_3\cdot\boldsymbol{a}_1 & \boldsymbol{a}_3\cdot\boldsymbol{a}_2 & \boldsymbol{a}_3\cdot\boldsymbol{a}_3 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So, together with (6.7) and that $\det\mathcal{F} > 0$

$$g_{ij} = p_{,i}\cdot p_{,j}, \qquad \nu = \boldsymbol{a}_3.$$

Then

$$h_{ij} = (\boldsymbol{a}_i)_{,j}\cdot\nu = p_{,ij}\cdot\nu,$$

that is, the coefficients of the second fundamental form coincides with (h_{ij}). □

Exercises

6-1 Let $\theta\colon U \to \mathbb{R}$ be a C^∞-function defined on a simply connected domain U of the uv-plane $\mathbb{R}^2$. Assuming θ satisfies $\theta_{uv} = \sin\theta$, prove that there exists a surface $p\colon U \to \mathbb{R}^3$ whose first and second fundamental forms are

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2, \qquad I\!I = 2\sin\theta\, du\, dv.$$

6-2 Let $\sigma\colon U \to \mathbb{R}$ be a C^∞-function defined on a simply connected domain U of the uv-plane $\mathbb{R}^2$. Assuming σ satisfies $\Delta\sigma = -\frac{1}{2}\sinh\sigma$, prove that there exists a surface $p\colon U \to \mathbb{R}^3$ with

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad I\!I = \frac{1}{2}\bigl((e^{2\sigma}+1)du^2 + (e^{2\sigma}-1)dv^2\bigr).$$