Introduction

This is a first half of two series of lectures, *Advanced Topics in Geometry A1* and *B1*, in which the fundamental theorem for surface theory and its applications are treated.

Throughout this lecture, object of our interest is “surfaces in Euclidean 3-space”. The goal is to give an comprehensive proof of the fundamental theorem for surface theory ([UY17, Theorem 17.2, see also Appendi B.10]). To accomplish the proof, mathematical tools including the theory of ordinary differential equations and the Frobenius intebrability theorem are expalined.

An aim of the lectures for students is to observe mathematical view around undergraduate calculus and linear algebra.

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1 Overview

Euclidean space

In this lecture, we denote by $\mathbb{R}^n$ the n-dimensional *Euclidean space* with canonical *inner product* $\langle\ ,\ \rangle$:

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \boldsymbol{x}^T\boldsymbol{y} = x_1y_1 + \dots + x_ny_n \qquad \text{for} \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n, \tag{1.1}$$

here, we regard an element of $\mathbb{R}^n$ as a column vector, and $(*)^T$ denotes the matrix transposition. Set[1]

$$\|\boldsymbol{x}\| := \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}, \qquad d(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{y} - \boldsymbol{x}\| \qquad (\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n) \tag{1.2}$$

which is called the *norm* of $\boldsymbol{x}$, and the *distance* of $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively.

A map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is called *isometry* if

$$d\bigl(f(\boldsymbol{x}), f(\boldsymbol{y})\bigr) = d(\boldsymbol{x}, \boldsymbol{y}) \tag{1.3}$$

holds for any $\boldsymbol{x}$ and $\boldsymbol{y} \in \mathbb{R}^n$.

Definition 1.1. An $n \times n$ real matrix R is said to be an *orthogonal matrix* if $R^TR = \mathrm{id}$ holds, where id is the $n \times n$ *identity matrix*.

The determinant of an orthogonal matrix R is 1 or -1. We denote by $\mathrm{O}(n)$ the set of $n \times n$ orthogonal matrices, and

$$\mathrm{SO}(n) := \{R \in \mathrm{O}(n)\,;\, \det R = 1\}. \tag{1.4}$$

Fact 1.2. *A map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is isometry if and only if it is written in the form*

$$f(\boldsymbol{x}) = R\boldsymbol{x} + \boldsymbol{a} \qquad \bigl(R \in \mathrm{O}(n), \boldsymbol{a} \in \mathbb{R}^n\bigr). \tag{1.5}$$

If R in (1.5) is a member of $\mathrm{SO}(n)$, f is said to be *orientation preserving*.

The Fundamental Theorem for surface Theory

Our object in this lecture is *surfaces* in Euclidean 3-space. The simplest question is:

Question 1.3. *What quantity determines a shape of surface?*

It is necessary for mathematical formulation of this question to express the surface. Among several ways to explain surfaces, we regard a surface as a *parametrization*, that is, a map [2]

$$\boldsymbol{f}\colon U \ni (u, v) \mapsto \boldsymbol{f}(u, v) \in \mathbb{R}^3,$$

where U is a *domain* [3]of $\mathbb{R}^2$.

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[1] "$A := B$" means that "A is defined by B".

[2] Unless confusion, points in the source domain are represented by row vectors.

[3] A domain is a connected open subset $U \subset \mathbb{R}^n$.

Example 1.4. • Set $U := (-\pi, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ and

$$\boldsymbol{f} : U \ni (u, v) \mapsto \boldsymbol{f}(u, v) = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ \sin v \end{pmatrix} \in \mathbb{R}^3$$

is a parametrization of the unit sphere in $\mathbb{R}^3$. The parameter u (resp. v) represents the longitude (resp. the latitude) of the point of the sphere.

• Set $V := (-\pi, \pi) \times \mathbb{R}$ and

$$\boldsymbol{g} : V \ni (s, t) \mapsto \boldsymbol{g}(s, t) = \begin{pmatrix} \cos s \operatorname{sech} t \\ \sin s \operatorname{sech} t \\ \tanh t \end{pmatrix} \in \mathbb{R}^3.$$

Then $\boldsymbol{g}$ parametrizes the unit sphere, and the st-plane is regarded as the Mercator's world map.

Then the following "fundamental theorem" is one of the answer:

Theorem (The Fundamental Theorem for surface theory)**.** *Let*

- $U \subset \mathbb{R}^2$ *be a simply connected domain,*
- I *be a positive definite symmetric quadratic form on* U
- II *be a symmetric quadratic form on* U*.*

Assume I *and* II *satisfy the Gauss and Codazzi equations. Then there exists a surface* $\boldsymbol{f} : U \to \mathbb{R}^3$ *whose first and second fundamental forms are* I *and* II*, respectively.*

Moreover, such an $\boldsymbol{f}$ *is unique up to orientation preserving isometry of* $\mathbb{R}^3$*.*

The undefined words in the statement, and mathematical meanings of the theorem will be explained through the lecture, and our goal is to prove this theorem.

Commutativity of partial derivatives

One of the most important fact in undergraduate calculus is the following "commutativity of partial derivatives".

Theorem 1.5. *Let* $f : U \to \mathbb{R}$ *be a function defined on a domain* U *of* $\mathbb{R}^2$ *and fix a point* $p = (u, v) \in U$*. If the second derivative* $\partial^2 f/(\partial x \partial y) = f_{yx}$ *and* $\partial^2 f/(\partial y \partial x) = f_{xy}$ *are both defined on* U *and continuous at* p*, then*

$$\frac{\partial^2 f}{\partial x \partial y}(p) = \frac{\partial^2 f}{\partial y \partial x}(p)$$

holds.

Proof. Take $(h, k) \in \mathbb{R}^2$ satisfying $(u + th, v + sk) \in U$ for all $t, s \in [0, 1]$. Let

$$g(h, k) := f(u + h, v + k) - f(u, v + k) - f(u + h, v) + f(u, v).$$

Since the partial derivative f_x exists on U, the function of one variable $F_1(t) := g(th, k)$ is differentiable on $0 \leqq t \leqq 1$. Then the mean value theorem implies that there exists $\theta_1 = \theta_1(h, k)$ with $0 < \theta_1 < 1$ such that

$$g(h, k) = F_1(1) = F_1(1) - F_1(0) = F_1'(\theta_1) = \big(f_x(u + \theta_1 h, v + k) - f_x(u + \theta_1 h, v)\big)h = F_2(1)h,$$

where $F_2(s) := f_x(u+\theta_1 h, v+sk) - f_x(u+\theta_1 h, v)$ $(0 \leqq s \leqq 1)$. Since $(f_x)_y$ exists on U, F_2 is differentiable on $0 \leqq s \leqq 1$. So, applying mean value theorem again, there exists $\theta_2 = \theta_2(h,k) \in (0,1)$ such that

$$F_2(1) = F_2'(\theta_2) = f_{xy}(u+\theta_1 h, v+\theta_2 k)k.$$

Summing up, there exists θ_1, $\theta_2 \in (0,1)$ depending on h and k such that

$$g(h,k) = f_{xy}(u+\theta_1 h, v+\theta_2 k)hk. \tag{1.6}$$

On the other hand, changing roles of h and k, we know that there exist φ_1, $\varphi_2 \in (0,1)$ such that

$$g(h,k) = f_{yx}(u+\varphi_1 h, v+\varphi_2 k)hk. \tag{1.7}$$

Then

$$f_{xy}(u+\theta_1 h, v+\theta_2 k) = f_{yx}(u+\varphi_1 h, v+\varphi_2 k)$$

whenever $hk \neq 0$. Here, taking limit $(h,k) \to (0,0)$, we have

$$(u+\theta_1 h, v+\theta_2 k) \to (u,v), \qquad (u+\varphi_1 h, v+\varphi_2 k) \to (u,v)$$

because θ_j, $\varphi_j \in (0,1)$ for $j = 1,2$. Thus, by continuity of f_{xy} and f_{yx}, we have $f_{xy}(u,v) = f_{yx}(u,v)$. □

Definition 1.6. A function f defined on a domain $U \subset \mathbb{R}^2$ is said to be

(1) *of class* C^0 if it is continuous on U,

(2) *of class* C^1 if there exists a partial derivative f_x and f_y on U, and both of them are continuous,

(3) *of class* C^r $(r = 2,3,\dots)$ if it is of class C^{r-1} and all of the $(r-1)$-st partial differentials are of class C^1, and

(4) *of class* C^∞ if it is of class C^r for arbitrary non-negative integer r.

Using these terms, we have

Corollary 1.7. *If a function $f\colon U \to \mathbb{R}$ defined on a domain U of $\mathbb{R}^2$ is of class C^2, then $f_{xy} = f_{yx}$ holds on U.*

In this lecture, functions are assumed to be of class C^∞. So partial differentials are always commutative.

Inverse of the commutativity—Poincaré lemma

A *differential* 1*-form*, or a 1*-form* defined on a domain $U \subset \mathbb{R}^2$ is the form

$$\alpha = a(x,y)\,dx + b(x,y)\,dy$$

where a and b are C^∞-functions defined on U. The *total differential*, or simply the *differential*, of C^∞-function f defined as

$$df := f_x\,dx + f_y\,dy$$

is a typical example of differential forms.

A *differential* 2*-form* is a form

$$\omega = c(x,y)\,dx \wedge dy$$

where c is a C^∞-function. The *exterior differential* $d\alpha$

$$d\alpha = d(a\,dx + b\,dy) = (b_x - a_y)\,dx \wedge dy$$

of 1-form $\alpha = a\,dx + b\,dy$ is a typical example.

Lemma 1.8. *Let f be a C^∞-function defined on a domain $U \subset \mathbb{R}^2$. Then $d(df) = 0$ holds.*

Proof. $d(df) = d(f_x\,dx + f_y\,dy) = (f_{yx} - f_{xy})\,dx \wedge dy = 0.$ □

Theorem 1.9 (Poincaré lemma)**.** *Let U be a simply connected domain, and α a differential 1-form defined on U. If $d\alpha = 0$, then there exists a C^∞ function f defined on U such that $df = \alpha$.*

The definition, fundamental properties of simple connectedness will be given in Section 3.

Exercises

1-1 Let $f(x,y) = e^{ax}\cos y$, where a is a constant. Find a function $g(x,y)$ satisfying

$$g_x = -f_y, \qquad g_y = f_x, \qquad g(0,0) = 0.$$

1-2 Let $U = \mathbb{R}^2 \setminus \{(t,0)\,;\, t \leqq 0\}$ and consider a 1-form

$$\alpha = a(x,y)\,dx + b(x,y)\,dy := \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

on U. Take a point $\mathrm{P} = (r\cos\theta, r\sin\theta) \in U$ ($r > 1$, $0 < \theta < \pi$), and two curves

$$\begin{aligned} c_1(t) &:= \bigl(x_1(t), y_1(t)\bigr) = (\cos t, \sin t) && (0 \leqq t \leqq \theta), \\ c_2(s) &:= \bigl(x_2(s), y_2(s)\bigr) = (s\cos\theta, s\sin\theta) && (1 \leqq s \leqq r), \end{aligned}$$

whose union gives a curve joining $(1,0)$ and P. Compute the line integral

$$\begin{aligned} \int_{c_1 \cup c_2} \alpha := &\int_0^\theta \left(a(x_1(t), y_1(t))\,\frac{dx_1}{dt}\,dt + b(x_1(t), y_1(t))\,\frac{dy_1}{dt}\,dt \right) \\ &+ \int_1^r \left(a(x_2(s), y_2(s))\,\frac{dx_2}{ds}\,ds + b(x_2(s), y_2(s))\,\frac{dy_2}{ds}\,ds \right). \end{aligned}$$

2 Ordinary Differential Equations

The fundamental theorem for ordinary differential equations.

Consider a function

(2.1) $$\boldsymbol{f}\colon I\times U\ni(t,\boldsymbol{x})\longmapsto\boldsymbol{f}(t,\boldsymbol{x})\in\mathbb{R}^m$$

of class C^1, where $I\subset\mathbb{R}$ is an interval and $U\subset\mathbb{R}^m$ is a domain in the Euclidean space $\mathbb{R}^m$. For any fixed $t_0\in I$ and $\boldsymbol{x}_0\in U$, the condition

(2.2) $$\frac{d}{dt}\boldsymbol{x}(t)=\boldsymbol{f}\bigl(t,\boldsymbol{x}(t)\bigr),\qquad \boldsymbol{x}(t_0)=\boldsymbol{x}_0$$

of an $\mathbb{R}^m$-valued function $t\mapsto\boldsymbol{x}(t)$ is called the *initial value problem of ordinary differential equation*, *ODE* for short, for unknown function $\boldsymbol{x}(t)$. For a subinterval J of I with $t_0\in I$, a function $\boldsymbol{x}\colon J\to U$ satisfying (2.2) is called a *solution* of the initial value problem.

Fact 2.1 (The existence theorem for ODE's)**.** *Let $\boldsymbol{f}\colon I\times U\to\mathbb{R}^m$ be a C^1-function as in* (2.1)*. Then, for any $\boldsymbol{x}_0\in U$ and $t_0\in I$, there exists a positive number ε and a C^1-function $\boldsymbol{x}\colon I\cap(t_0-\varepsilon,t_0+\varepsilon)\to U$ satisfying* (2.2)*.*

Take two solutions $\boldsymbol{x}_j\colon J_j\to U$ $(j=1,2)$ of (2.2) defined on subintervals $J_j\subset I$ containing t_0. Then the function $\boldsymbol{x}_2$ is said to be an *extension* of $\boldsymbol{x}_1$ if $J_1\subset J_2$ and $\boldsymbol{x}_2(t)=\boldsymbol{x}_1(t)$ for all $t\in J_1$. A solution $\boldsymbol{x}$ of (2.2) is said to be *maximal* if there are no non-trivial extension of it.

Fact 2.2 (The uniqueness for ODE's)**.** *The maximal solution of* (2.2) *is unique.*

Fact 2.3 (Smoothness of the solutions)**.** *If $\boldsymbol{f}\colon I\times U\to\mathbb{R}^m$ is of class C^r $(r=1,\dots,\infty)$, the solution of* (2.2) *is of class C^{r+1}. Here, $\infty+1=\infty$, as a convention.*

Let $V\subset\mathbb{R}^k$ be another domain of $\mathbb{R}^k$ and consider a C^∞-function

(2.3) $$\boldsymbol{h}\colon I\times U\times V\ni(t,\boldsymbol{x};\boldsymbol{\alpha})\mapsto\boldsymbol{h}(t,\boldsymbol{x};\boldsymbol{\alpha})\in\mathbb{R}^m.$$

For fixed $t_0\in I$, we denote by $\boldsymbol{x}(t;\boldsymbol{x}_0,\boldsymbol{\alpha})$ the (unique, maximal) solution of (2.2) for $\boldsymbol{f}(t,\boldsymbol{x})=\boldsymbol{h}(t,\boldsymbol{x};\boldsymbol{\alpha})$. Then

Fact 2.4. *The map $(t,\boldsymbol{x}_0;\boldsymbol{\alpha})\mapsto\boldsymbol{x}(t;\boldsymbol{x}_0,\boldsymbol{\alpha})$ is of class C^∞.*

Example 2.5. (1) Let $m=1$, $I=\mathbb{R}$, $U=\mathbb{R}$ and $f(t,x)=\lambda x$, where λ is a constant. Then $x(t)=x_0\exp(\lambda t)$ defined on $\mathbb{R}$ is the maximal solution to

$$\frac{d}{dt}x(t)=f(t,x(t))=\lambda x(t),\qquad x(0)=x_0.$$

(2) Let $m=2$, $I=\mathbb{R}$, $U=\mathbb{R}^2$ and $\boldsymbol{f}(t;(x,y))=(y,-\omega^2x)$, where ω is a constant. Then

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=\begin{pmatrix}x_0\cos\omega t+\frac{y_0}{\omega}\sin\omega t\\-x_0\omega\sin\omega t+y_0\cos\omega t\end{pmatrix}$$

is the unique solution of

$$\frac{d}{dt}\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=\begin{pmatrix}y(t)\\-\omega^2x(t)\end{pmatrix},\qquad\begin{pmatrix}x(0)\\y(0)\end{pmatrix}=\begin{pmatrix}x_0\\y_0\end{pmatrix},$$

defined on $\mathbb{R}$. This equation can be considered as a single equation

$$\frac{d^2}{dt^2}x(t)=-\omega^2x(t),\quad x(0)=x_0,\quad\frac{dx}{dt}(0)=y_0$$

of order 2.

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(3) Let $m=1$, $I=\mathbb{R}$, $U=\mathbb{R}$ and $f(t,x)=t(1+x^2)$. Then $x(t)=\tan\frac{t^2}{2}$ defined on $(-\sqrt{\pi},\sqrt{\pi})$ is the unique maximal solution of the initial value problem

$$\frac{dx}{dt}=t(1+x^2), \qquad x(0)=0.$$

Linear Ordinary Differential Equations.

The ordinary differential equation (2.2) is said to be *linear* if the function (2.1) is a linear function in $\boldsymbol{x}$, that is, a linear differential equation is in a form

$$\frac{d}{dt}\boldsymbol{x}(t)=A(t)\boldsymbol{x}(t)+\boldsymbol{b}(t),$$

where $A(t)$ and $\boldsymbol{b}(t)$ are $m\times m$-matrix-valued and $\mathbb{R}^m$-valued functions in t, respectively.

For the sake of later use, we consider, in this lecture, the special form of linear differential equation for matrix-valued unknown functions as follows: Let $\mathrm{M}_n(\mathbb{R})$ be the set of $n\times n$-matrices with real components, and take functions

$$\Omega\colon I\longrightarrow \mathrm{M}_n(\mathbb{R}), \quad \text{and} \qquad B\colon I\longrightarrow \mathrm{M}_n(\mathbb{R}),$$

where $I\subset\mathbb{R}$ is an interval. Identifying $\mathrm{M}_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$, we assume Ω and B are continuous functions (with respect to the topology of $\mathbb{R}^{n^2}=\mathrm{M}_n(\mathbb{R})$). Then we can consider the linear ordinary differential equation for matrix-valued unknown $X(t)$ as

$$\frac{dX(t)}{dt}=X(t)\Omega(t)+B(t), \qquad X(t_0)=X_0, \tag{2.4}$$

where X_0 is given constant matrix.

Then, the fundamental theorem of *linear* ordinary equation states that *the maximal solution of* (2.4) *is defined on whole* I. To prove this, we prepare some materials related to matrix-valued functions.

Preliminaries: Matrix Norms.

Denote by $\mathrm{M}_n(\mathbb{R})$ the set of $n\times n$-matrices with real components, which can be identified the vector space $\mathbb{R}^{n^2}$. In particular, the Euclidean norm of $\mathbb{R}^{n^2}$ induces a norm

$$|X|_{\mathrm{E}}=\sqrt{\operatorname{tr}(X^TX)}=\sqrt{\sum_{i,j=1}^{n}x_{ij}^2} \tag{2.5}$$

on $\mathrm{M}_n(\mathbb{R})$. On the other hand, we let

$$|X|_{\mathrm{M}}:=\sup\left\{\frac{|X\boldsymbol{v}|}{|\boldsymbol{v}|}\,;\,\boldsymbol{v}\in\mathbb{R}^n\setminus\{\boldsymbol{0}\}\right\}, \tag{2.6}$$

where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^n$.

Lemma 2.6. (1) *The map* $X\mapsto|X|_{\mathrm{M}}$ *is a norm of* $\mathrm{M}_n(\mathbb{R})$.

(2) *For* X, $Y\in\mathrm{M}_n(\mathbb{R})$, *it holds that* $|XY|_{\mathrm{M}}\leqq|X|_{\mathrm{M}}\,|Y|_{\mathrm{M}}$.

(3) *Let* $\lambda=\lambda(X)$ *be the maximum eigenvalue of semi-positive definite symmetric matrix* X^TX. *Then* $|X|_{\mathrm{M}}=\sqrt{\lambda}$ *holds.*

(4) $(1/\sqrt{n})|X|_{\mathrm{E}}\leqq|X|_{\mathrm{M}}\leqq|X|_{\mathrm{E}}$.

(5) *The map* $|\cdot|_{\mathrm{M}}\colon \mathrm{M}_n(\mathbb{R})\to\mathbb{R}$ *is continuous with respect to the Euclidean norm.*

Proof. Since $|X\boldsymbol{v}|/|\boldsymbol{v}|$ is invariant under scalar multiplications to $\boldsymbol{v}$, we have $|X|_{\mathrm{M}}=\sup\{|X\boldsymbol{v}|\,;\,\boldsymbol{v}\in S^{n-1}\}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Since $S^{n-1}\ni\boldsymbol{x}\mapsto|A\boldsymbol{x}|\in\mathbb{R}$ is a continuous function defined on a compact space, it takes the maximum. Thus, the right-hand side of (2.6) is well-defined. It is easy to verify that $|\cdot|_{\mathrm{M}}$ satisfies the axiom of the norm[4].

Since $A:=X^TX$ is positive semi-definite, its eigenvalues λ_j $(j=1,\dots,n)$ are non-negative real numbers. In particular, there exists an orthonormal basis $[\boldsymbol{a}_j]$ of $\mathbb{R}^n$ satisfying $A\boldsymbol{a}_j=\lambda_j\boldsymbol{a}_j$ $(j=1,\dots,n)$. Let λ be the maximum eigenvalue of A, and write $\boldsymbol{v}=v_1\boldsymbol{a}_1+\dots+v_n\boldsymbol{a}_n$. Then it holds that

$$\langle X\boldsymbol{v},X\boldsymbol{v}\rangle=\lambda_1v_1^2+\dots+\lambda_nv_n^2\leqq\lambda\,\langle\boldsymbol{v},\boldsymbol{v}\rangle\,,$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product of $\mathbb{R}^n$. The equality of this inequality holds if and only if $\boldsymbol{v}$ is the λ-eigenvector, proving (3). Noticing that the norm (2.5) is invariant under conjugations $X\mapsto P^TXP$ $(P\in\mathrm{O}(n))$, we obtain $|X|_{\mathrm{E}}=\sqrt{\lambda_1+\dots+\lambda_n}$ by diagonalizing X^TX by an orthogonal matrix P. Then we obtain (4). Hence two norms $|\cdot|_{\mathrm{E}}$ and $|\cdot|_{\mathrm{M}}$ induce the same topology as $\mathrm{M}_n(\mathbb{R})$. In particular, we have (5). □

Preliminaries: Matrix-valued Functions.

Lemma 2.7. *Let X and Y be C^∞-maps defined on a domain $U\subset\mathbb{R}^m$ into $\mathrm{M}_n(\mathbb{R})$. Then*

(1) $\dfrac{\partial}{\partial u_j}(XY)=\dfrac{\partial X}{\partial u_j}Y+X\dfrac{\partial Y}{\partial u_j}$,

(2) $\dfrac{\partial}{\partial u_j}\det X=\operatorname{tr}\left(\widetilde{X}\dfrac{\partial X}{\partial u_j}\right)$, *and*

(3) $\dfrac{\partial}{\partial u_j}X^{-1}=-X^{-1}\dfrac{\partial X}{\partial u_j}X^{-1}$,

where $\widetilde{X}$ is the cofactor matrix of X, and we assume in (3) *that X is a regular matrix.*

Proof. The formula (1) holds because the definition of matrix multiplication and the Leibniz rule, Denoting $'=\partial/\partial u_j$,

$$O=(\mathrm{id})'=(X^{-1}X)'=(X^{-1})X'+(X^{-1})'X$$

implies (3), where id is the identity matrix.

Decompose the matrix X into column vectors as $X=(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)$. Since the determinant is multi-linear form for n-tuple of column vectors, it holds that

$$(\det X)'=\det(\boldsymbol{x}_1',\boldsymbol{x}_2,\dots,\boldsymbol{x}_n)+\det(\boldsymbol{x}_1,\boldsymbol{x}_2',\dots,\boldsymbol{x}_n)+\dots+\det(\boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_n').$$

Then by cofactor expansion of the right-hand side, we obtain (2). □

Proposition 2.8. *Assume two C^∞ matrix-valued functions $X(t)$ and $\Omega(t)$ satisfy*

$$\frac{dX(t)}{dt}=X(t)\Omega(t),\qquad X(t_0)=X_0. \tag{2.7}$$

Then

$$\det X(t)=(\det X_0)\exp\int_{t_0}^t\operatorname{tr}\Omega(\tau)\,d\tau \tag{2.8}$$

holds. In particular, if $X_0\in\mathrm{GL}(n,\mathbb{R})$,[5] *then $X(t)\in\mathrm{GL}(n,\mathbb{R})$ for all t.*

[4] $|X|_{\mathrm{M}}>0$ whenever $X\neq O$, $|\alpha X|_{\mathrm{M}}=|\alpha|\,|X|_{\mathrm{M}}$, and the triangle inequality $|X+Y|_{\mathrm{M}}\leqq|X|_{\mathrm{M}}+|Y|_{\mathrm{M}}$.
[5] $\mathrm{GL}(n,\mathbb{R})=\{A\in\mathrm{M}_n(\mathbb{R})\,;\,\det A\neq 0\}$: the general linear group.

Proof. By (2) of Lemma 2.7, we have

$$\frac{d}{dt}\det X(t) = \operatorname{tr}\left(\widetilde{X}(t)\frac{dX(t)}{dt}\right) = \operatorname{tr}\left(\widetilde{X}(t)X(t)\Omega(t)\right)$$
$$= \operatorname{tr}\bigl(\det X(t)\Omega(t)\bigr) = \det X(t)\operatorname{tr}\Omega(t).$$

Here, we used the relation $\widetilde{X}X = X\widetilde{X} = (\det X)\,\mathrm{id}$. Hence $\frac{d}{dt}\bigl(\rho(t)^{-1}\det X(t)\bigr) = 0$, where $\rho(t)$ is the right-hand side of (2.8). □

Corollary 2.9. *If $\Omega(t)$ in (2.7) satisfies $\operatorname{tr}\Omega(t) = 0$, then $\det X(t)$ is constant. In particular, if $X_0 \in \mathrm{SL}(n,\mathbb{R})$, X is a function valued in $\mathrm{SL}(n,\mathbb{R})$* [6].

Proposition 2.10. *Assume $\Omega(t)$ in (2.7) is skew-symmetric for all t, that is, $\Omega^T + \Omega$ is identically O. If $X_0 \in \mathrm{O}(n)$ (resp. $X_0 \in \mathrm{SO}(n)$)*[7]*, then $X(t) \in \mathrm{O}(n)$ (resp. $X(t) \in \mathrm{SO}(n)$) for all t.*

Proof. By (1) in Lemma 2.7,

$$\frac{d}{dt}(XX^T) = \frac{dX}{dt}X^T + X\left(\frac{dX}{dt}\right)^T$$
$$= X\Omega X^T + X\Omega^T X^T = X(\Omega + \Omega^T)X^T = O.$$

Hence XX^T is constant, that is, if $X_0 \in \mathrm{O}(n)$,

$$X(t)X(t)^T = X(t_0)X(t_0)^T = X_0X_0^T = \mathrm{id}\,.$$

If $X_0 \in \mathrm{O}(n)$, this proves the first case of the proposition. Since $\det A = \pm 1$ when $A \in \mathrm{O}(n)$, the second case follows by continuity of $\det X(t)$. □

Preliminaries: Norms of Matrix-Valued functions.

Let $I = [a,b]$ be a closed interval, and denote by $C^0(I, \mathrm{M}_n(\mathbb{R}))$ the set of continuous functions $X\colon I \to \mathrm{M}_n(\mathbb{R})$. For any positive number k, we define

$$||X||_{I,k} := \sup\left\{e^{-kt}|X(t)|_{\mathrm{M}}\,;\, t \in I\right\} \tag{2.9}$$

for $X \in C^0(I, \mathrm{M}_n(\mathbb{R}))$. When $k = 0$, $||\cdot||_{I,0}$ is the *uniform norm* for continuous functions, which is complete. Similarly, one can prove the following in the same way:

Lemma 2.11. *The norm $||\cdot||_{I,k}$ on $C^0(I, \mathrm{M}_n(\mathbb{R}))$ is complete.*

Linear Ordinary Differential Equations.

We prove the fundamental theorem for *linear* ordinary differential equations.

Proposition 2.12. *Let $\Omega(t)$ be a C^∞-function valued in $\mathrm{M}_n(\mathbb{R})$ defined on an interval I. Then for each $t_0 \in I$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,\mathrm{id}}(t)$ such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = \mathrm{id}\,. \tag{2.10}$$

[6] $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\, \det A = 1\}$; the special linear group.
[7] $\mathrm{O}(n) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\, A^TA = AA^T = \mathrm{id}\}$: the orthogonal group; $\mathrm{SO}(n) = \{A \in \mathrm{O}(n)\,;\, \det A = 1\}$: the special orthogonal group.

Proof. Uniqueness: Assume $X(t)$ and $Y(t)$ satisfy (2.10). Then

$$Y(t)-X(t)=\int_{t_0}^{t}\bigl(Y'(\tau)-X'(\tau)\bigr)\,d\tau=\int_{t_0}^{t}\bigl(Y(\tau)-X(\tau)\bigr)\Omega(\tau)\,d\tau \qquad \left({}'=\frac{d}{dt}\right)$$

holds. Take an arbitrary closed interval $J\subset I$. Then for an arbitrary $t\in J$,

$$\begin{aligned}|Y(t)-X(t)|_{\mathrm{M}} &\leqq \left|\int_{t_0}^{t}\left|\bigl(Y(\tau)-X(\tau)\bigr)\Omega(\tau)\right|_{\mathrm{M}}d\tau\right| \leqq \left|\int_{t_0}^{t}|Y(\tau)-X(\tau)|_{\mathrm{M}}\,|\Omega(\tau)|_{\mathrm{M}}\,d\tau\right|\\ &=\left|\int_{t_0}^{t}e^{-k\tau}\,|Y(\tau)-X(\tau)|_{\mathrm{M}}\,e^{k\tau}\,|\Omega(\tau)|_{\mathrm{M}}\,d\tau\right| \leqq ||Y-X||_{J,k}\sup_J|\Omega|_{\mathrm{M}}\left|\int_{t_0}^{t}e^{k\tau}\,d\tau\right|\\ &=||Y-X||_{J,k}\frac{\sup_J|\Omega|_{\mathrm{M}}}{|k|}e^{kt}\left|1-e^{-k(t-t_0)}\right| \leqq ||Y-X||_{J,k}\sup_J|\Omega|_{\mathrm{M}}\frac{e^{kt}}{|k|}\end{aligned}$$

holds, and hence

$$e^{-kt}|Y(t)-X(t)|_{\mathrm{M}}\leqq\frac{\sup_J|\Omega|_{\mathrm{M}}}{|k|}||Y-X||_{J,k}.$$

Thus, for an appropriate choice of $k\in\mathbb{R}$, it holds that

$$||Y-X||_{J,k}\leqq\frac{1}{2}||Y-X||_{J,k},$$

that is, $||Y-X||_{J,k}=0$, proving $Y(t)=X(t)$ for $t\in J$. Since J is arbitrary, $Y=X$ holds on I.
Existence: Take $a>t_0$ such that $J:=[t_0,a]\subset I$, and define a sequence $\{X_j\}$ of matrix-valued functions defined on I satisfying $X_0(t)=\mathrm{id}$ and

$$X_{j+1}(t)=\mathrm{id}+\int_{t_0}^{t}X_j(\tau)\Omega(\tau)\,d\tau \quad (j=0,1,2,\dots). \tag{2.11}$$

Then

$$\begin{aligned}|X_{j+1}(t)-X_j(t)|_{\mathrm{M}} &\leqq \int_{t_0}^{t}|X_j(\tau)-X_{j-1}(\tau)|_{\mathrm{M}}|\Omega(\tau)|_{\mathrm{M}}\,d\tau\\ &\leqq \frac{e^{kt)}}{|k|}\sup_J|\Omega|_{\mathrm{M}}||X_j-X_{j-1}||_{J,k},\end{aligned}$$

and hence $||X_{j+1}-X_j||_{J,k}\leqq\frac{1}{2}||X_j-X_{j-1}||_{J,k}$, for an appropriate choice of $k\in\mathbb{R}$, that is, $\{X_j\}$ is a Cauchy sequence with respect to $||\cdot||_{J,k}$. Thus, by completeness (Lemma 2.11), it converges to some $X\in C^0(J,\mathrm{M}_n(\mathbb{R}))$. By (2.11), the limit X satisfies

$$X(t_0)=\mathrm{id},\qquad X(t)=\mathrm{id}+\int_{t_0}^{t}X(\tau)\Omega(\tau)\,d\tau.$$

Applying the fundamental theorem of calculus, we can see that X satisfies $X'(t)=X(t)\Omega(t)$ ($'=d/dt$). By the same argument for $a<t_0$ with $J=[a,t_0]$, existence of the solution on I is proven.

Finally, we shall prove that X is of class C^∞. Since $X'(t)=X(t)\Omega(t)$, the derivative X' of X is continuous. Hence X is of class C^1, and so is $X(t)\Omega(t)$. Thus we have that $X'(t)$ is of class C^1, and then X is of class C^2. Iterating this argument, we can prove that $X(t)$ is of class C^r for arbitrary r. □

Corollary 2.13. *Let $\Omega(t)$ be a matrix-valued C^∞-function defined on an interval I. Then for each $t_0\in I$ and $X_0\in\mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X_{t_0,X_0}(t)$ defined on I such that*

$$\frac{dX(t)}{dt}=X(t)\Omega(t),\quad X(t_0)=X_0 \quad \bigl(X(t):=X_{t_0,X_0}(t)\bigr) \tag{2.12}$$

In particular, $X_{t_0,X_0}(t)$ is of class C^∞ in X_0 and t.

Proof. We rewrite $X(t)$ in Proposition 2.12 as $Y(t) = X_{t_0,\mathrm{id}}(t)$. Then the function

$$X(t) := X_0 Y(t) = X_0 X_{t_0,\mathrm{id}}(t), \tag{2.13}$$

is desired one. Conversely, assume $X(t)$ satisfies the conclusion. Noticing $Y(t)$ is a regular matrix for all t because of Proposition 2.8,

$$W(t) := X(t)Y(t)^{-1}$$

satisfies

$$\frac{dW}{dt} = \frac{dX}{dt}Y^{-1} - XY^{-1}\frac{dY}{dt}Y^{-1} = X\Omega Y^{-1} - XY^{-1}Y\Omega Y^{-1} = O,$$

that is, W is constant, and hence

$$W(t) = W(t_0) = X(t_0)Y(t_0)^{-1} = X_0.$$

So the uniqueness is obtained. The final part is obvious by the expression (2.13). □

Proposition 2.14. *Let $\Omega(t)$ and $B(t)$ be matrix-valued C^∞-functions defined on I. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function defined on I satisfying*

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0. \tag{2.14}$$

Proof. Rewrite X in Proposition 2.12 as $Y := X_{t_0,\mathrm{id}}$. Then

$$X(t) := \left(X_0 + \int_{t_0}^{t} B(\tau)Y^{-1}(\tau)\,d\tau\right)Y(t) \tag{2.15}$$

satisfies (2.14). Conversely, if X satisfies (2.14), $W := XY^{-1}$ satisfies

$$X' = W'Y + WY' = W'Y + WY\Omega, \quad X\Omega + B = WY\Omega + B,$$

and then we have $W' = BY^{-1}$. Since $W(t_0) = X_0$,

$$W = X_0 + \int_{t_0}^{t} B(\tau)Y^{-1}(\tau)\,d\tau.$$

Thus we obtain (2.15). □

Theorem 2.15. *Let I and U be an interval and a domain in $\mathbb{R}^m$, respectively, and let $\Omega(t,\boldsymbol{\alpha})$ and $B(t,\boldsymbol{\alpha})$ be matrix-valued C^∞-functions defined on $I \times U$ ($\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$). Then for each $t_0 \in I$, $\boldsymbol{\alpha} \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,X_0,\boldsymbol{\alpha}}(t)$ defined on I such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t,\boldsymbol{\alpha}) + B(t,\boldsymbol{\alpha}), \qquad X(t_0) = X_0. \tag{2.16}$$

Moreover,

$$I \times I \times \mathrm{M}_n(\mathbb{R}) \times U \ni (t, t_0, X_0, \boldsymbol{\alpha}) \mapsto X_{t_0,X_0,\boldsymbol{\alpha}}(t) \in \mathrm{M}_n(\mathbb{R})$$

is a C^∞-map.

Proof. Let $\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) := \Omega(t+t_0,\boldsymbol{\alpha})$ and $\widetilde{B}(t,\tilde{\boldsymbol{\alpha}}) = B(t+t_0,\boldsymbol{\alpha})$, and let $\widetilde{X}(t) := X(t+t_0)$. Then (2.16) is equivalent to

$$\frac{d\widetilde{X}(t)}{dt} = \widetilde{X}(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) + \widetilde{B}(t,\tilde{\boldsymbol{\alpha}}), \quad \widetilde{X}(0) = X_0, \tag{2.17}$$

where $\tilde{\boldsymbol{\alpha}} := (t_0, \alpha_1, \dots, \alpha_m)$. There exists the unique solution $\widetilde{X}(t) = \widetilde{X}_{0,X_0,\tilde{\boldsymbol{\alpha}}}(t)$ of (2.17) for each $\tilde{\boldsymbol{\alpha}}$ because of Proposition 2.14. So it is sufficient to show differentiability with respect to the parameter $\tilde{\boldsymbol{\alpha}}$. We set $Z = Z(t)$ the unique solution of

$$\frac{dZ}{dt} = Z\widetilde{\Omega} + \widetilde{X}\frac{\partial\widetilde{\Omega}}{\partial\alpha_j} + \frac{\partial\widetilde{B}}{\partial\alpha_j}, \qquad Z(0) = O. \tag{2.18}$$

Then it holds that $Z = \partial\widetilde{X}/\partial\alpha_j$. In particular, by the proof of Proposition 2.14, it holds that

$$Z = \frac{\partial\widetilde{X}}{\partial\alpha_j} = \left(\int_0^t \left(\widetilde{X}(\tau)\frac{\partial\widetilde{\Omega}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j} + \frac{\partial\widetilde{B}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j}\right) Y^{-1}(\tau)d\tau\right) Y(t).$$

Here, $Y(t)$ is the unique matrix-valued C^∞-function satisfying $Y'(t) = Y(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}})$, and $Y(0) = \mathrm{id}$. Hence $\widetilde{X}$ is a C^∞-function in $(t,\tilde{\boldsymbol{\alpha}})$. □

An Application: Fundamental Theorem for Space Curves.

A C^∞-map $\gamma\colon I \to \mathbb{R}^3$ defined on an interval $I \subset \mathbb{R}$ into $\mathbb{R}^3$ is said to be a *regular curve* if $\dot{\gamma} \neq \mathbf{0}$ holds on I. For a regular curve $\gamma(t)$, there exists a parameter change $t = t(s)$ such that $\tilde{\gamma}(s) := \gamma(t(s))$ satisfies $|\tilde{\gamma}'(s)| = 1$. Such a parameter s is called the *arc-length parameter*.

Let $\gamma(s)$ be a regular curve in $\mathbb{R}^3$ parametrized by the arc-length satisfying $\gamma''(s) \neq \mathbf{0}$ for all s. Then

$$\boldsymbol{e}(s) := \gamma'(s), \qquad \boldsymbol{n}(s) := \frac{\gamma''(s)}{|\gamma''(s)|}, \qquad \boldsymbol{b}(s) := \boldsymbol{e}(s) \times \boldsymbol{n}(s)$$

forms a positively oriented orthonormal basis $\{\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{b}\}$ of $\mathbb{R}^3$ for each s. Regarding each vector as column vector, we have the matrix-valued function

$$\mathcal{F}(s) := (\boldsymbol{e}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)) \in \mathrm{SO}(3). \tag{2.19}$$

in s, which is called the *Frenet frame* associated to the curve γ. Under the situation above, we set

$$\kappa(s) := |\gamma''(s)| > 0, \qquad \tau(s) := -\left\langle \boldsymbol{b}'(s), \boldsymbol{n}(s) \right\rangle,$$

which are called the *curvature* and *torsion*, respectively, of γ. Using these quantities, the Frenet frame satisfies

$$\frac{d\mathcal{F}}{ds} = \mathcal{F}\Omega, \qquad \Omega = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}. \tag{2.20}$$

Proposition 2.16. *The curvature and the torsion are invariant under the transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ of $\mathbb{R}^3$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$). Conversely, two curves $\gamma_1(s)$, $\gamma_2(s)$ parametrized by arc-length parameter have common curvature and torsion, there exist $A \in \mathrm{SO}(3)$ and $\boldsymbol{b} \in \mathbb{R}^3$ such that $\gamma_2 = A\gamma_1 + \boldsymbol{b}$.*

Proof. Let κ, τ and $\mathcal{F}_1$ be the curvature, torsion and the Frenet frame of γ_1, respectively. Then the Frenet frame of $\gamma_2 = A\gamma_1 + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) is $\mathcal{F}_2 = A\mathcal{F}_1$. Hence both $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy (2.20), and then γ_1 and γ_2 have common curvature and torsion.

Conversely, assume γ_1 and γ_2 have common curvature and torsion. Then the frenet frame $\mathcal{F}_1$, $\mathcal{F}_2$ both satisfy (2.20). Let $\mathcal{F}$ be the unique solution of (2.20) with $\mathcal{F}(t_0) = \mathrm{id}$. Then by the proof of Corollary 2.13, we have $\mathcal{F}_j(t) = \mathcal{F}_j(t_0)\mathcal{F}(t)$ $(j = 1, 2)$. In particular, since $\mathcal{F}_j \in \mathrm{SO}(3)$, $\mathcal{F}_2(t) = A\mathcal{F}_1(t)$ $(A := \mathcal{F}_2(t_0)\mathcal{F}_1(t_0)^{-1} \in \mathrm{SO}(3))$. Comparing the first column of these, $\gamma_2'(s) = A\gamma_1'(t)$ holds. Integrating this, the conclusion follows. □

Theorem 2.17 (The fundamental theorem for space curves)**.**
Let $\kappa(s)$ and $\tau(s)$ be C^∞-functions defined on an interval I satisfying $\kappa(s) > 0$ on I. Then there exists a space curve $\gamma(s)$ parametrized by arc-length whose curvature and torsion are κ and τ, respectively. Moreover, such a curve is unique up to transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ $(A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3)$ of $\mathbb{R}^3$.

Proof. We have already shown the uniqueness in Proposition 2.16. We shall prove the existence: Let $\Omega(s)$ be as in (2.20), and $\mathcal{F}(s)$ the solution of (2.20) with $\mathcal{F}(s_0) = \mathrm{id}$. Since Ω is skew-symmetric, $\mathcal{F}(s) \in \mathrm{SO}(3)$ by Proposition 2.10. Denoting the column vectors of $\mathcal{F}$ by $\boldsymbol{e}$, $\boldsymbol{n}$, $\boldsymbol{b}$, and let

$$\gamma(s) := \int_{s_0}^{s} \boldsymbol{e}(\sigma)\, d\sigma.$$

Then $\mathcal{F}$ is the Frenet frame of γ, and κ, and τ are the curvature and torsion of γ, respectively. □

Exercises

2-1 Find the maximal solution of the initial value problem

$$\frac{dx}{dt} = x(1-x), \qquad x(0) = a,$$

where a is a real number.

2-2 Let $x = x(t)$ be the maximal solution of an initial value problem of differential equation

$$\frac{d^2x}{dt^2} = -\sin x, \qquad x(0) = 0, \quad \frac{dx}{dt}(0) = 2.$$

- Show that $\dfrac{dx}{dt} = 2\cos\dfrac{x}{2}$.
- Verify that x is defined on $\mathbb{R}$, and compute $\lim_{t\to\pm\infty} x(t)$.

2-3 Find an explicit expression of a space curve $\gamma(s)$ parametrized by the arc-length s, whose curvature κ and torsion τ satisfy

$$\kappa = \tau = \frac{1}{\sqrt{2}(1+s^2)}.$$

3 Integrability Conditions

Let $U \subset \mathbb{R}^m$ be a domain of $(\mathbb{R}^m; u^1, \dots, u^m)$ and consider an m-tuple of $n \times n$-matrix valued C^∞-maps

$$\Omega_j \colon \mathbb{R}^m \supset U \longrightarrow \mathrm{M}_n(\mathbb{R}) \qquad (j = 1, \dots, m). \tag{3.1}$$

In this section, we consider an initial value problem of a system of linear partial differential equations

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0, \tag{3.2}$$

where $\mathrm{P}_0 = (u_0^1, \dots, u_0^m) \in U$ is a fixed point, X is an $n \times n$-matrix valued unknown, and $X_0 \in \mathrm{M}_n(\mathbb{R})$.

Proposition 3.1. *If a C^∞-map $X \colon U \to \mathrm{M}_n(\mathbb{R})$ defined on a domain $U \subset \mathbb{R}^m$ satisfies* (3.2) *with $X_0 \in \mathrm{GL}(n, \mathbb{R})$, then $X(\mathrm{P}) \in \mathrm{GL}(n, \mathbb{R})$ for all $\mathrm{P} \in U$. In addition, if Ω_j $(j = 1, \dots, m)$ are skew-symmetric and $X_0 \in \mathrm{SO}(n)$, then $X(\mathrm{P}) \in \mathrm{SO}(n)$ holds for all $\mathrm{P} \in U$.*

Proof. Since U is connected, there exists a continuous path $\gamma_0 \colon [0, 1] \to U$ such that $\gamma_0(0) = \mathrm{P}_0$ and $\gamma_0(1) = \mathrm{P}$. By Whitney's approximation theorem (cf. Theorem 6.21 in [Lee13]), there exists a smooth path $\gamma \colon [0, 1] \to U$ joining P_0 and P approximating γ_0. Since $\hat{X} := X \circ \gamma$ satisfies (3.4) with $\hat{X}(0) = X_0$, Proposition 2.8 yields that $\det \hat{X}(1) \neq 0$ whenever $\det X_0 \neq 0$. Moreover, if Ω_j's are skew-symmetric, so is $\Omega_\gamma(t)$ in (3.4). Thus, by Proposition 2.10, we obtain the latter half of the proposition. □

Proposition 3.2. *If a matrix-valued C^∞ function $X \colon U \to \mathrm{GL}(n, \mathbb{R})$ satisfies* (3.2)*, it holds that*

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j \Omega_k - \Omega_k \Omega_j \tag{3.3}$$

for each (j, k) with $1 \leqq j < k \leqq m$.

Proof. Differentiating (3.2) by u^k, we have

$$\frac{\partial^2 X}{\partial u^k \partial u^j} = \frac{\partial X}{\partial u^k}\Omega_j + X\frac{\partial \Omega_j}{\partial u^k} = X\left(\frac{\partial \Omega_j}{\partial u^k} + \Omega_k \Omega_j\right).$$

On the other hand, switching the roles of j and k, we get

$$\frac{\partial^2 X}{\partial u^j \partial u^k} = X\left(\frac{\partial \Omega_k}{\partial u^j} + \Omega_j \Omega_k\right).$$

Since X is of class C^∞, the left-hand sides of these equalities coincide, and so are the right-hand sides. Since $X \in \mathrm{GL}(n, \mathbb{R})$, the conclusion follows. □

The equality (3.3) is called the *integrability condition* or *compatibility condition* of (3.2).
The chain rule yields the following:

Lemma 3.3. *Let $X \colon U \to \mathrm{M}_n(\mathbb{R})$ be a C^∞-map satisfying* (3.2)*. Then for each smooth path $\gamma \colon I \to U$ defined on an interval $I \subset \mathbb{R}$, $\hat{X} := X \circ \gamma : I \to \mathrm{M}_n(\mathbb{R})$ satisfies the ordinary differential equation*

$$\frac{d\hat{X}}{dt}(t) = \hat{X}(t)\Omega_\gamma(t) \qquad \left(\Omega_\gamma(t) := \sum_{j=1}^m \Omega_j \circ \gamma(t) \frac{du^j}{dt}(t)\right) \tag{3.4}$$

on I, where $\gamma(t) = \bigl(u^1(t), \dots, u^m(t)\bigr)$.

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Lemma 3.4. *Let $\mathit{\Omega}_j \colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ be C^∞-maps defined on a domain $U \subset \mathbb{R}^m$ which satisfy* (3.3)*. Then for each smooth map*

$$\sigma \colon D \ni (t, w) \longmapsto \sigma(t, w) = (u^1(t, w), \dots, u^m(t, w)) \in U$$

defined on a domain $D \subset \mathbb{R}^2$, it holds that

$$\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} - TW + WT = 0, \tag{3.5}$$

where

$$T := \sum_{j=1}^{m} \widetilde{\mathit{\Omega}}_j \frac{\partial u^j}{\partial t}, \quad W := \sum_{j=1}^{m} \widetilde{\mathit{\Omega}}_j \frac{\partial u^j}{\partial w} \quad (\widetilde{\mathit{\Omega}}_j := \mathit{\Omega}_j \circ \sigma). \tag{3.6}$$

Proof. By the chain rule, we have

$$\begin{aligned}
\frac{\partial T}{\partial w} &= \sum_{j,k=1}^{m} \frac{\partial \mathit{\Omega}_j}{\partial u^k} \frac{\partial u^k}{\partial w} \frac{\partial u^j}{\partial t} + \sum_{j=1}^{m} \widetilde{\mathit{\Omega}}_j \frac{\partial^2 u^j}{\partial w \partial t}, \\
\frac{\partial W}{\partial t} &= \sum_{j,k=1}^{m} \frac{\partial \mathit{\Omega}_j}{\partial u^k} \frac{\partial u^k}{\partial t} \frac{\partial u^j}{\partial w} + \sum_{j=1}^{m} \widetilde{\mathit{\Omega}}_j \frac{\partial^2 u^j}{\partial t \partial w} \\
&= \sum_{j,k=1}^{m} \frac{\partial \mathit{\Omega}_k}{\partial u^j} \frac{\partial u^j}{\partial t} \frac{\partial u^k}{\partial w} + \sum_{j=1}^{m} \widetilde{\mathit{\Omega}}_j \frac{\partial^2 u^j}{\partial t \partial w}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} &= \sum_{j,k=1}^{m} \left(\frac{\partial \mathit{\Omega}_j}{\partial u^k} - \frac{\partial \mathit{\Omega}_k}{\partial u^j} \right) \frac{\partial u^k}{\partial w} \frac{\partial u^j}{\partial t} \\
&= \sum_{j,k=1}^{m} \left(\widetilde{\mathit{\Omega}}_j \widetilde{\mathit{\Omega}}_k - \widetilde{\mathit{\Omega}}_k \widetilde{\mathit{\Omega}}_j \right) \frac{\partial u^k}{\partial w} \frac{\partial u^j}{\partial t} \\
&= \left(\sum_{j=1}^{m} \widetilde{\mathit{\Omega}}_j \frac{\partial u^j}{\partial t} \right) \left(\sum_{k=1}^{m} \widetilde{\mathit{\Omega}}_k \frac{\partial u^k}{\partial w} \right) - \left(\sum_{k=1}^{m} \widetilde{\mathit{\Omega}}_k \frac{\partial u^k}{\partial w} \right) \left(\sum_{j=1}^{m} \widetilde{\mathit{\Omega}}_j \frac{\partial u^j}{\partial t} \right) \\
&= TW - WT.
\end{aligned}$$

Thus (3.5) holds.

Integrability of linear systems. The main theorem in this section is the following theorem:

Theorem 3.5. *Let $\mathit{\Omega}_j \colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ be C^∞-functions defined on a simply connected domain $U \subset \mathbb{R}^m$ satisfying* (3.3)*. Then for each $\mathrm{P}_0 \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique $n \times n$-matrix valued function $X \colon U \to \mathrm{M}_n(\mathbb{R})$ satisfying* (3.2)*. Moreover,*

- *if $X_0 \in \mathrm{GL}(n, \mathbb{R})$, $X(\mathrm{P}) \in \mathrm{GL}(n, \mathbb{R})$ holds on U,*
- *if $X_0 \in \mathrm{SO}(n)$ and $\mathit{\Omega}_j$ $(j = 1, \dots, m)$ are skew-symmetric matrices, $X \in \mathrm{SO}(n)$ holds on U.*

Proof. The latter half is a direct conclusion of Proposition 3.1. We show the existence of X: Take a smooth path $\gamma \colon [0, 1] \to U$ joining P_0 and P. Then by Theorem 2.15, there exists a unique C^∞-map $\hat{X} \colon [0, 1] \to \mathrm{M}_n(\mathbb{R})$ satisfying (3.4) with initial condition $\hat{X}(0) = X_0$.

We shall show that the value $\hat{X}(1)$ does not depend on choice of paths joining P_0 and P. To show this, choose another smooth path $\tilde{\gamma}$ joining P_0 and P. Since U is simply connected, there exists a homotopy between γ and $\tilde{\gamma}$, that is, there exists a continuous map $\sigma_0\colon [0,1]\times[0,1]\ni (t,w)\mapsto \sigma_0(t,w)\in U$ satisfying

$$\begin{aligned} \sigma_0(t,0)&=\gamma(t), &\qquad \sigma_0(t,1)&=\tilde{\gamma}(t),\\ \sigma_0(0,w)&=\mathrm{P}_0, &\qquad \sigma_0(1,w)&=\mathrm{P}. \end{aligned} \tag{3.7}$$

Then, by Whitney's approximation theorem (Theorem 6.21 in [Lee13]) again, there exists a smooth map $\sigma\colon [0,1]\times[0,1]\to U$ satisfying the same boundary conditions as (3.7):

$$\begin{aligned} \sigma(t,0)&=\gamma(t), &\qquad \sigma(t,1)&=\tilde{\gamma}(t),\\ \sigma(0,w)&=\mathrm{P}_0, &\qquad \sigma(1,w)&=\mathrm{P}. \end{aligned} \tag{3.8}$$

We set T and W as in (3.6). For each fixed $w\in[0,1]$, there exists $X_w\colon[0,1]\to \mathrm{M}_n(\mathbb{R})$ such that

$$\frac{dX_w}{dt}(t)=X_w(t)T(t,w),\qquad X_w(0)=X_0.$$

Since $T(t,w)$ is smooth in t and w, the map

$$\check{X}\colon [0,1]\times[0,1]\ni(t,w)\mapsto X_w(t)\in \mathrm{M}_n(\mathbb{R})$$

is a smooth map, because of smoothness in parameter α in Theorem 2.15. To show that $\hat{X}(1)=\check{X}(1,0)$ does not depend on choice of paths, it is sufficient to show that

$$\frac{\partial\check{X}}{\partial w}=\check{X}W \tag{3.9}$$

holds on $[0,1]\times[0,1]$. In fact, by (3.8), $W(1,w)=0$ for all $w\in[0,1]$, and then (3.9) implies that $\check{X}(1,w)$ is constant.

We prove (3.9): By definition, it holds that

$$\frac{\partial\check{X}}{\partial t}=\check{X}T,\qquad \check{X}(0,w)=X_0 \tag{3.10}$$

for each $w\in[0,1]$. Hence by (3.5),

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial\check{X}}{\partial w}&=\frac{\partial^2\check{X}}{\partial t\partial w}=\frac{\partial^2\check{X}}{\partial w\partial t}=\frac{\partial}{\partial w}(\check{X}T)\\ &=\frac{\partial\check{X}}{\partial w}T+\check{X}\frac{\partial T}{\partial w}=\frac{\partial\check{X}}{\partial w}T+\check{X}\left(\frac{\partial W}{\partial t}+TW-WT\right)\\ &=\frac{\partial\check{X}}{\partial w}T+\check{X}\frac{\partial W}{\partial t}+\frac{\partial\check{X}}{\partial t}W-\check{X}WT\\ &=\frac{\partial}{\partial t}(\check{X}W)+\left(\frac{\partial\check{X}}{\partial w}-\check{X}W\right)T. \end{aligned}$$

So, the function $Y_w(t):=\partial\check{X}/\partial w-\check{X}W$ satisfies the ordinary differential equation

$$\frac{dY_w}{dt}(t)=Y_w(t)T(t,w),\quad Y_w(0)=O$$

for each $w\in[0,1]$. Thus, by the uniqueness of the solution, $Y_w(t)=O$ holds on $[0,1]\times[0,1]$. Hence we have (3.9).

Thus, $\hat{X}(1)$ depends only on the end point P of the path. Hence we can set $X(\mathrm{P}):=\hat{X}(1)$ for each $\mathrm{P}\in U$, and obtain a map $X\colon U\to\mathrm{M}_n(\mathbb{R})$. Finally we show that X is the desired solution. The initial condition $X(\mathrm{P}_0)=X_0$ is obviously satisfied. On the other hand, if we set

$$Z(\delta):=X(u^1,\dots,u^j+\delta,\dots,u^m),$$

$Z(\delta)$ satisfies the equation (3.4) for the path $\gamma(\delta) := (u^1, \dots, u^j + \delta, \dots, u^m)$ with $Z(0) = X(\mathrm{P})$. Since $\Omega_\gamma = \Omega_j$,

$$\frac{\partial X}{\partial u^j}(\mathrm{P}) = \left.\frac{dZ}{d\delta}\right|_{\delta=0} = Z(0)\Omega_j(\mathrm{P}) = X(\mathrm{P})\Omega_j(\mathrm{P})$$

which completes the proof. □

Application: Poincaré's lemma.

Theorem 3.6 (Poincaré's lemma)**.** *If a differential* 1*-form*

$$\omega = \sum_{j=1}^{m} \alpha_j(u^1, \dots, u^m)\, du^j$$

defined on a simply connected domain $U \subset \mathbb{R}^m$ *is closed, that is,* $d\omega = 0$ *holds, then there exists a* C^∞*-function* f *on* U *such that* $df = \omega$*. Such a function* f *is unique up to additive constants.*

Proof. Since

$$d\omega = \sum_{i<j} \left(\frac{\partial \alpha_j}{\partial u^i} - \frac{\partial \alpha_i}{\partial u^j} \right) du^i \wedge du^j,$$

the assumption is equivalent to

$$\frac{\partial \alpha_j}{\partial u^i} - \frac{\partial \alpha_i}{\partial u^j} = 0 \qquad (1 \leqq i < j \leqq m). \tag{3.11}$$

Consider a system of linear partial differential equations with unknown ξ, a 1×1-matrix valued function (i.e. a real-valued function), as

$$\frac{\partial \xi}{\partial u^j} = \xi \alpha_j \quad (j = 1, \dots, m), \qquad \xi(u_0^1, \dots, u_0^m) = 1. \tag{3.12}$$

Then it satisfies (3.3) because of (3.11). Hence by Theorem 3.5, there exists a smooth function $\xi(u^1, \dots, u^m)$ satisfying (3.12). In particular, Proposition 2.8 yields $\xi = \det \xi$ never vanishes. Hence $\xi(u_0^1, \dots, u_0^m) = 1 > 0$ means that $\xi > 0$ holds on U. Letting $f := \log \xi$, we have the function f satisfying $df = \omega$.

Next, we show the uniqueness: if two functions f and g satisfy $df = dg = \omega$, it holds that $d(f - g) = 0$. Hence by connectivity of U, $f - g$ must be constant. □

Application: Conjugation of Harmonic functions. In this paragraph, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. It is well-known that a smooth function

$$f\colon U \ni u + \mathrm{i}\, v \longmapsto \xi(u,v) + \mathrm{i}\, \eta(u,v) \in \mathbb{C} \qquad (\mathrm{i} = \sqrt{-1}) \tag{3.13}$$

defined on a domain $U \subset \mathbb{C}$ is *holomorphic* if and only if it satisfies the following relation, called the *Cauchy-Riemann equations*:

$$\frac{\partial \xi}{\partial u} = \frac{\partial \eta}{\partial v}, \qquad \frac{\partial \xi}{\partial v} = -\frac{\partial \eta}{\partial u}. \tag{3.14}$$

Definition 3.7. A function $f\colon U \to \mathbb{R}$ defined on a domain $U \subset \mathbb{R}^2$ is said to be *harmonic* if it satisfies

$$\Delta f = f_{uu} + f_{vv} = 0.$$

The operator Δ is called the *Laplacian*.

Proposition 3.8. *If function f in (3.13) is holomorphic, $\xi(u,v)$ and $\eta(u,v)$ are harmonic functions.*

Proof. By (3.14), we have

$$\xi_{uu} = (\xi_u)_u = (\eta_v)_u = \eta_{vu} = \eta_{uv} = (\eta_u)_v = (-\xi_v)_v = -\xi_{vv}.$$

Hence $\Delta\xi = 0$. Similarly,

$$\eta_{uu} = (-\xi_v)_u = -\xi_{vu} = -\xi_{uv} = -(\xi_u)_v = -(\eta_v)_v = -\eta_{vv}.$$

Thus $\Delta\eta = 0$. □

Theorem 3.9. *Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u,v)$ a C^∞-function harmonic on U*[8] *Then there exists a C^∞ harmonic function η on U such that $\xi(u,v) + \mathrm{i}\,\eta(u,v)$ is holomorphic on U.*

Proof. Let $\alpha := -\xi_v\,du + \xi_u\,dv$. Then by the assumption,

$$d\alpha = (\xi_{vv} + \xi_{uu})\,du \wedge dv = 0$$

holds, that is, α is a closed 1-form. Hence by simple connectivity of U and the Poincaré's lemma (Theorem 1.9), there exists a function η such that $d\eta = \eta_u\,du + \eta_v\,dv = \alpha$. Such a function η satisfies (3.14) for given ξ. Hence $\xi + \mathrm{i}\,\eta$ is holomorphic in $u + \mathrm{i}\,v$. □

Example 3.10. A function $\xi(u,v) = e^u \cos v$ is harmonic. Set

$$\alpha := -\xi_v\,du + \xi_u\,dv = e^u \sin v\,du + e^u \cos v\,dv.$$

Then $\eta(u,v) = e^u \sin v$ satisfies $d\eta = \alpha$. Hence

$$\xi + \mathrm{i}\,\eta = e^u(\cos v + \mathrm{i}\sin v) = e^{u+\mathrm{i}\,v}$$

is holomorphic in $u + \mathrm{i}\,v$.

Definition 3.11. The harmonic function η in Theorem 3.9 is called the *conjugate* harmonic function of ξ.

[8] The theorem holds under the assumption of C^2-differentiability.

Exercises

3-1 Let

$$\xi_1(u,v) := \frac{u}{u^2+v^2}, \qquad \xi_2(u,v) := \log\sqrt{u^2+v^2}$$

be functions defined on non-simply connected domain $U := \mathbb{R}^2 \setminus \{(0,0)\}$.

(1) Show that both ξ_1 and ξ_2 are harmonic on U.

(2) Verify that there exists a conjugate harmonic function η_1 of ξ_1 on U.

(3) Prove that there exists no conjugate harmonic function η_2 of ξ_2 on U.

3-2 Consider a linear system of partial differential equationcs for 3×3-matrix valued unknown X on a domain $U \subset \mathbb{R}^2$ as

$$\frac{\partial X}{\partial u} = X\Omega, \quad \frac{\partial X}{\partial v} = X\Lambda, \qquad \left(\Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix}\right),$$

where (u,v) are the canonical coordinate system of $\mathbb{R}^2$, and α, β and h_j^i $(i,j=1,2)$ are smooth functions defined on U. Write down the integrability conditions in terms of α, β and h_j^i.

4 A review of surface theory

In this section, we review the classical surface theory in the Euclidean 3-space. The textbook [UY17] is one of the fundamental references of this material.

4.1 Preliminaries

Euclidean space Let $\mathbb{R}^3$ be the Euclidean 3-space, that is, the 3-dimensional affine space $\mathbb{R}^3$ endowed with the Euclidean *inner product* "$\cdot$", where[9]

$$\boldsymbol{x}\cdot\boldsymbol{y} := \boldsymbol{x}^T\boldsymbol{y} = x^1y^1 + x^2y^2 + x^3y^3, \qquad \text{where} \quad \boldsymbol{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} \in \mathbb{R}^3. \tag{4.1}$$

The Euclidean *norm* $|\ |$ and the Euclidean *distance* $d(\ ,\ )$ is defined as

$$|\boldsymbol{x}| := \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}} = \sqrt{(x^1)^2+(x^2)^2+(x^3)^2}, \qquad d(\boldsymbol{x},\boldsymbol{y}) = |\boldsymbol{y}-\boldsymbol{x}| \qquad (\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3). \tag{4.2}$$

A map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is called an *isometry* if it preserves the distance function d: $d\bigl(f(\boldsymbol{x}), f(\boldsymbol{y})\bigr) = d(\boldsymbol{x},\boldsymbol{y})$ ($\boldsymbol{x}$, $\boldsymbol{y}\in\mathbb{R}^3$).

Fact 4.1. *A map $f\colon \mathbb{R}^3\to\mathbb{R}^3$ is an isometry if and only if f is in a form*

$$f(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b} \qquad (A\in \mathrm{O}(3), \boldsymbol{b}\in\mathbb{R}^3), \tag{4.3}$$

where $\mathrm{O}(3)$ *is the set of* 3×3 *orthogonal matrices.*

An isometry in (4.3) is said to be *orientation preserving* if $A\in\mathrm{SO}(3)$, that is, A is an orthogonal matrix with $\det A = 1$.

The *outer product* or *vector product* $\boldsymbol{x}\times\boldsymbol{y}$ of $\boldsymbol{x}$, $\boldsymbol{y}\in\mathbb{R}^3$ is defined by

$$\det(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = (\boldsymbol{x}\times\boldsymbol{y})\cdot\boldsymbol{z}. \tag{4.4}$$

Immersed surfaces Let $U\subset\mathbb{R}^2$ be a domain of the uv-plane $\mathbb{R}^2$. A C^∞-map $p\colon U\to\mathbb{R}^3$ is called an *immersion* or a *parametrization of a regular surface* if

$$p_u(u,v) := \frac{\partial p}{\partial u}(u,v), \quad \text{and} \quad p_v(u,v) := \frac{\partial p}{\partial v}(u,v) \qquad \text{are linearly independent} \tag{4.5}$$

at each point $(u,v)\in U$. The *unit normal vector field* to an immersion $p\colon U\to\mathbb{R}^3$ is a C^∞-map $\nu\colon U\to\mathbb{R}^3$ satisfying

$$\nu\cdot p_u = \nu\cdot p_v = 0, \qquad |\nu| = 1 \tag{4.6}$$

for each point on U.

The *first fundamental form* ds^2 is defined by

$$ds^2 := dp\cdot dp = E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad \bigl(E := p_u\cdot p_u, F := p_u\cdot p_v = p_v\cdot p_u, G := p_v\cdot p_v\bigr), \tag{4.7}$$

where the subscript u (resp. v) means the partial derivative with respect to the variable u (resp. v). The three functions E, F and G defined on U are called the coefficients of the first fundamental form.

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[9] According to a traditional manner, the indices of coordinate functions are written as superscripts.

Similarly, taking account of the identity

$$\nu_u \cdot p_v = (\nu \cdot p_v)_u - \nu \cdot p_{vu} = 0 - \nu \cdot p_{vu} = -\nu \cdot p_{uv} = \nu_v \cdot p_u,$$

we define the *second fundamental form* as

$$II := -d\nu \cdot dp = L\,du^2 + 2M\,du\,dv + N\,dv^2, \qquad \big(L := -p_u \cdot \nu_u, M := -p_u \cdot \nu_v = -p_v \cdot \nu_u, N := -p_v \cdot \nu_v\big). \tag{4.8}$$

The symmetric matrices

$$\widehat{I} := \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (p_u, p_v), \qquad \widehat{II} := \begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (\nu_u, \nu_v)$$

are called the first and second fundamental matrices, respectively.

By the Cauchy-Schwarz inequality, it holds that

$$EG - F^2 = |p_u|^2 |p_v|^2 - (p_u \cdot p_v)^2 > 0,$$

and then the first fundamental matrix $\widehat{I}$ is a regular matrix. The *area element* of the surface is defined as

$$d\mathcal{A} := \sqrt{EG - F^2}\,du\,dv. \tag{4.9}$$

In fact, the area of a part of surface corresponding to a relatively compact domain $\Omega \subset U$ is computed as

$$\mathcal{A}(\Omega) := \iint_\Omega d\mathcal{A} = \iint_\Omega \sqrt{EG - F^2}\,du\,dv.$$

Since $\widehat{I}$ is regular, the matrix

$$A := \widehat{I}^{-1}\,\widehat{II} = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix}, \tag{4.10}$$

called the *Weingarten matrix*, is defined. It is known that the eigenvalues λ_1 and λ_2 of A are real numbers, and called the *principal curvatures*. The *Gaussian curvature* K and the *mean curvature* H are defined as

$$K := \lambda_1 \lambda_2 = \det A = \frac{\det \widehat{II}}{\det \widehat{I}}, \qquad H := \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}\operatorname{tr} A. \tag{4.11}$$

4.2 Gauss frames

To simplify computations, and for a future generalization for higher dimensional case, we switch the notation here to the “index” style. Write the coordinate system of $U \subset \mathbb{R}^2$ by (u^1, u^2) instead of (u, v), and denote

$$f_{,1} = \frac{\partial f}{\partial u^1}, \qquad f_{,2} = \frac{\partial f}{\partial u^2},$$

that is, the subscript number following a comma means the partial derivative with respect to the corresponding variable. Using these notations, the first fundamental form is expressed as

$$ds^2 = dp \cdot dp = \sum_{i,j=1}^{2} g_{ij}\,du^i\,du^j, \qquad (g_{ij} := p_{,i} \cdot p_{,j}). \tag{4.12}$$

Similarly, the second fundamental form is written as

$$II = -dp \cdot d\nu = \sum_{i,j=1}^{2} h_{ij}\, du^i\, du^j, \qquad (h_{ij} := -p_{,i} \cdot \nu_{,j} = -p_{,j} \cdot \nu_{,i} = p_{,ij} \cdot \nu). \tag{4.13}$$

Since the first fundamental matrix $\widehat{I} = (g_{ij})_{i,j=1,2}$ has positive determinant, its inverse matrix exists. We denote the component of the inverse by $\widehat{I}^{\,-1} = (g^{ij})$, using superscripts instead of subscripts. By definition, it holds that

$$g^{ij} = g^{ji} \qquad \text{and} \qquad \sum_{k=1}^{2} g^{ik} g_{kj} = \delta^i_j = \begin{cases} 1 & (i = j) \\ 0 & (\text{otherwise}), \end{cases} \tag{4.14}$$

where δ stands for *Kronecker's delta symbol*. Using these, the Weingarten matrix A as in (4.10) and the Gaussian curvature K in (4.11) are expressed as

$$A = (A^i_j), \qquad A^i_j = \sum_{k=1}^{2} g^{ik} h_{kj}, \qquad K = \det A = \frac{\det(h_{ij})}{\det(g_{ij})}. \tag{4.15}$$

Since p is an immersion, $\{p_{,1}(u^1,u^2), p_{,2}(u^1,u^2), \nu(u^1,u^2)\}$ are linearly independent for each point $(u^1,u^2) \in U$. Hence we obtain a smooth map

$$\mathcal{F}\colon U \ni (u^1,u^2) \mapsto \bigl(p_{,1}(u^1,u^2), p_{,2}(u^1,u^2), \nu(u^1,u^2)\bigr) \in \mathrm{GL}(3,\mathbb{R}), \tag{4.16}$$

where $\mathrm{GL}(3,\mathbb{R})$ is the set of 3×3 regular matrices with real components. The map $\mathcal{F}$ is called the *Gauss frame* of the surface p.

Theorem 4.2. *The Gauss frame $\mathcal{F}$ satisfies*

$$\frac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j \qquad \left(\Omega_j := \begin{pmatrix} \Gamma^1_{1j} & \Gamma^1_{2j} & -A^1_j \\ \Gamma^2_{1j} & \Gamma^2_{2j} & -A^2_j \\ h_{1j} & h_{2j} & 0 \end{pmatrix}\right) \qquad (j = 1,2), \tag{4.17}$$

where h_{ij}'s are the coefficients of the second fundamental form, A^i_j's are the components of the Weingarten matrix, and

$$\Gamma^k_{ij} := \frac{1}{2}\sum_{l=1}^{2} g^{kl}(g_{il,j} + g_{lj,i} - g_{ij,l}), \qquad (i,j,k = 1,2) \tag{4.18}$$

The functions Γ^k_{ij} in (4.18) are called the *Christoffel symbols*, and the equation (4.17) is called the *Gauss-Weingarten formula*. By decomposing $\mathcal{F}$ into columns, the Gauss-Weingarten formula is restated as

$$p_{,ij} = \left(\sum_{l=1}^{2} \Gamma^l_{ij} p_{,l}\right) + h_{ij}\nu, \tag{4.19}$$

$$\nu_{,j} = -\sum_{l=1}^{2} A^l_j p_{,l}. \tag{4.20}$$

The equality (4.19) and (4.20) are called the *Gauss formula* and *Weingarten formula*, respectively.

Proof of Theorem 4.2. Since $\{p_{,1}, p_{,2}, \nu\}$ is a basis of $\mathbb{R}^3$ at each point $(u^1,u^2) \in U$, the second derivative $p_{,ij}$ is expressed as a linear combination of $\{p_{,1}, p_{,2}, \nu\}$:

$$p_{,ij} = \Lambda^1_{ij} p_{,1} + \Lambda^2_{ij} p_{,2} + \eta_{ij}\nu = \left(\sum_{l=1}^{2} \Lambda^l_{ij} p_{,l}\right) + \eta_{ij}\nu, \tag{4.21}$$

where Λ_{ij}^l and η_{ij} are smooth functions in (u^1, u^2). Since ν is perpendicular to $p_{,l}$, (4.13) implies

$$\eta_{ij} = p_{,ij} \cdot \nu = h_{ij}.$$

On the other hand, taking inner product with $p_{,k}$, we have

$$p_{,ij} \cdot p_{,k} = \sum_{l=1}^{2} \Lambda_{ij}^l p_{,l} \cdot p_{,k} = \sum_{l=1}^{2} g_{lk} \Lambda_{ij}^l. \tag{4.22}$$

Here, by the Leibniz rule, the left-hand side is computed as

$$\begin{aligned} p_{,ij} \cdot p_{,k} &= (p_{,i} \cdot p_{,k})_{,j} - p_{,i} \cdot p_{,kj} = g_{ik,j} - (p_{,i} \cdot p_{,j})_{,k} + p_{,ik} \cdot p_{,j} \\ &= g_{ik,j} - g_{ij,k} + (p_{,k} \cdot p_{,j})_{,i} - p_{,ij} \cdot p_{,k} = g_{ik,j} - g_{ij,k} + g_{jk,i} - p_{,ij} \cdot p_{,k}, \end{aligned}$$

and thus, $p_{,ij} \cdot p_{,k} = \frac{1}{2}(g_{ik,j} + g_{kj,i} - g_{ij,k})$. Then (4.22) turns to be

$$\frac{1}{2}(g_{ik,j} + g_{kj,i} - g_{ij,k}) = p_{,ij} \cdot p_{,k} = \sum_{l=1}^{2} g_{lk} \Lambda_{ij}^l.$$

Multiplying g^{sk} on the both sides of the equality above, and summing up it over $k = 1$ and 2, we have

$$\frac{1}{2} \sum_{k=1}^{2} g^{sk}(g_{ik,j} + g_{kj,i} - g_{ij,k}) = \sum_{k=1}^{2} \sum_{l=1}^{2} g^{sk} g_{lk} \Lambda_{ij}^l = \sum_{l=1}^{2} \sum_{s=1}^{2} g^{sk} g_{kl} \Lambda_{ij}^l = \sum_{l=1}^{2} \delta_l^s \Lambda_{ij}^l = \Lambda_{ij}^s.$$

This implies that Λ_{ij}^l coincides with the Christoffel symbol (4.18). Summing up, the Gauss formula (4.19) is proven.

Next, we prove the Weingarten formula: Since $\nu \cdot \nu = 1$, $\nu_{,j}$ is perpendicular to ν. Hence we can write

$$\nu_{,j} = \sum_{l=1}^{2} B_j^l p_{,l},$$

and then by (4.21),

$$-h_{ij} = p_{,i} \cdot \nu_{,j} = \sum_{l=1}^{2} B_j^l p_{,l} \cdot p_{,i} = \sum_{l=1}^{2} g_{il} B_j^l.$$

So,

$$B_j^k = \sum_{l=1}^{2} \delta_l^k B_j^l = \sum_{l=1}^{2} \sum_{s=1}^{2} g^{ks} g_{sl} B_j^l = -\sum_{s=1}^{2} g^{ks} h_{js} = -A_j^k,$$

proving (4.20). □

For later use, we prepare the following formulas on the Christoffel symbols:

Proposition 4.3. *The Christoffel symbol in* (4.18) *satisfies*

$$\Gamma_{ij}^k = \Gamma_{ji}^k \tag{4.23}$$

$$g_{ij,k} = \sum_{l=1}^{2} (g_{lj} \Gamma_{ik}^l + g_{il} \Gamma_{kj}^l), \tag{4.24}$$

$$\frac{\partial g}{\partial u^i} = 2g \sum_{l=1}^{2} \Gamma_{il}^l, \qquad (g := \det \widehat{I} = g_{11} g_{22} - g_{12}^2), \tag{4.25}$$

where the indices i, j *and* k *run over* 1 *and* 2.

Proof. Since

$$p_{,ij} = \Gamma_{ij}^1 p_{,1} + \Gamma_{ij}^2 p_{,2} + h_{ij}\nu \quad \text{and} \quad p_{,ji} = \Gamma_{ji}^1 p_{,1} + \Gamma_{ji}^2 p_{,2} + h_{ji}\nu,$$

(4.23) follows.

The second formula (4.24) is obtained as

$$\begin{aligned} g_{ij,k} &= \left(p_{,i}\cdot p_{,j}\right)_{,k} = p_{,ik}\cdot p_{,j} + p_{,i}\cdot p_{,jk} \\ &= \left(\sum_{l=1}^2 \Gamma_{ik}^l (p_{,l}\cdot p_{,j}) + h_{ik}(\nu\cdot p_{,j})\right) + \left(\sum_{l=1}^2 \Gamma_{jk}^l (p_{,i}\cdot p_{,l}) + h_{jk}(p_{,i}\cdot\nu)\right) \\ &= \sum_{l=1}^2 \left(g_{lj}\Gamma_{ik}^l + g_{il}\Gamma_{kj}^l\right). \end{aligned}$$

Finally, differentiating $g = \det \widehat{I}$,

$$\begin{aligned} \frac{\partial g}{\partial u^i} &= \operatorname{tr}\left(\widetilde{\widehat{I}}\frac{\partial \widehat{I}}{\partial u^i}\right) = (\det \widehat{I})\operatorname{tr}\left(\widehat{I}^{-1}\widehat{I}_{,i}\right) = g\sum_{l,m=1}^2 g^{lm} g_{lm,i} \\ &= g\sum_{l,m,s=1}^2 g^{lm}\left(g_{ms}\Gamma_{li}^s + g_{ls}\Gamma_{im}^s\right) = g\left(\sum_{l,s=1}^2 \delta_s^l \Gamma_{li}^s + \sum_{m,s=1}^2 \delta_s^m \Gamma_{im}^s\right) \\ &= g\left(\sum_{l=1}^2 \Gamma_{li}^l + \sum_{m=1}^2 \Gamma_{im}^m\right) = 2g\sum_{l=1}^2 \Gamma_{il}^l, \end{aligned}$$

where $\widetilde{\widehat{I}} = (\det \widehat{I})\,\widehat{I}^{-1}$ is the cofactor matrix of $\widehat{I}$. Thus we have (4.25). □

4.3 Orthonormal frames

The Gauss and Weingarten formulas (Theorem 4.2) are the fundamental equations which express how the fundamental forms determine shape of surfaces. In this section, another formulation of Gauss-Weingarten formulas using orthonormal frames is given. In this subsection, we write the coordinate system of $\mathbb{R}^2$ by (u, v), again.

Adapted frames

Let $p\colon U \to \mathbb{R}^3$ be an immersion of a domain $U \subset \mathbb{R}^2$ into the Euclidean 3-space, and take the unit normal vector field $\nu\colon U \to \mathbb{R}^3$ of p. For simplicity, we assume that ν is compatible to the canonical orientation of U, that is, $\det \mathcal{F} = \det(p_u, p_v, \nu) > 0$, where $\mathcal{F}$ is the Gauss frame.

Definition 4.4. A C^∞-map $\mathcal{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) : U \to \mathrm{SO}(3)$ is called an *adapted* (orthonormal) frame of the surface $p\colon U \to \mathbb{R}^3$ if $\boldsymbol{e}_3$ coincides with the unit normal vector field ν.

Example 4.5. Let $p\colon \mathbb{R}^2 \supset U \ni (u, v) \mapsto p(u, v) \in \mathbb{R}^3$ be an immersion and let ν be the unit normal vector field of p which is compatible to the orientation of U. We let

$$\boldsymbol{e}_1^0 := \frac{1}{\sqrt{E}} p_u, \qquad \boldsymbol{e}_2^0 := \frac{1}{\sqrt{E}\sqrt{EG - F^2}}(E p_v - F p_u),$$

where E, F, G are the coefficients of the first fundamental form as in (4.7). Since $\nu := \boldsymbol{e}_3^0$ is perpendicular to both p_u and p_v, $\mathcal{E}^0 := (\boldsymbol{e}_1^0, \boldsymbol{e}_2^0, \boldsymbol{e}_3^0)$ is an adapted frame of p. Remark that $\{\boldsymbol{e}_1^0, \boldsymbol{e}_2^0\}$ is an orthonormal frame of the orthogonal complement of ν (that is, the tangent plane) obtained by applying the Gram-Schmidt orthogonalization to (p_u, p_v).

Gauge transformations

An adapted frame has an ambiguity of a rotation of the frame $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ of the tangent plane. In fact, for an arbitrary function $\phi\colon U \to \mathbb{R}$,

$$\widetilde{\mathcal{E}} = \mathcal{E}R, \qquad R := R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.26}$$

is another adapted frame. Conversely, we have the following:

Lemma 4.6. *Let $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ be adapted frames of the surface $p\colon U \to \mathbb{R}^3$, where U is a simply connected domain. Then there exists a function $\phi\colon U \to \mathbb{R}$ satisfying* (4.26)*.*

Proof. Since $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are valued in SO(3) with common third columns, an SO(3)-valued function $R := \mathcal{E}^{-1}\widetilde{\mathcal{E}}$ is expressed as

$$R = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} R_0 & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \qquad \left(R_0 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : U \to \mathrm{SO}(2) \right),$$

where a and b are C^∞-functions defined on U. Fix a point $(u_0, v_0) \in U$. Since $R_0 \in \mathrm{SO}(2)$, $a^2 + b^2 = 1$, and then there exists an angle ϕ_0 such that

$$a(u_0, v_0) = \cos\phi_0, \quad b(u_0, v_0) = \sin\phi_0. \tag{4.27}$$

Consider a differential 1-form

$$\omega := -b\,da + a\,db = (-ba_u + ab_u)\,du + (-ba_v + ab_v)\,dv.$$

Then

$$d\omega = \big((-ba_v + ab_v)_u - (-ba_u + ab_u)_v\big)\,du \wedge dv = 2(a_u b_v - b_u a_v)\,du \wedge dv.$$

On the other hand, differentiating $a^2 + b^2 = 1$, it holds that

$$0 = a\,da + b\,db = (aa_u + bb_u)du + (aa_v + bb_v)dv, \qquad \text{that is,} \qquad aa_u = -bb_u, \qquad aa_v = -bb_v.$$

Hence

$$\begin{aligned} a d\omega &= 2(aa_u b_v - b_u aa_v)\,du \wedge dv = 2(-bb_u b_v + b_u aa_v)\,du \wedge dv = 0, \\ b d\omega &= 2(a_u bb_v - bb_u a_v)\,du \wedge dv = 2(-a_u aa_v + aa_u a_v) du \wedge dv = 0, \end{aligned}$$

which implies that $d\omega = 0$ because $(a, b) \neq (0, 0)$. Then by the Poincaré lemma (Theorem 1.9), there exists the unique function $\phi\colon U \to \mathbb{R}$ such that

$$d\phi = \omega = -b\,da + a\,db, \qquad \phi(u_0, v_0) = \phi_0. \tag{4.28}$$

Set $\tilde{a} := \cos\phi$ and $\tilde{b} := \sin\phi$. Then by (4.28), both R_0 and

$$\hat{R}_0 = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

satisfies the same systems of differential equations

$$X_u = X \begin{pmatrix} 0 & -\phi_u \\ \phi_u & 0 \end{pmatrix}, \qquad X_v = X \begin{pmatrix} 0 & -\phi_v \\ \phi_v & 0 \end{pmatrix}$$

with the same initial condition. Hence $R_0 = \hat{R}_0$, which is the conclusion. □

A transformation of adapted frames as in Lemma 4.6 is called a *gauge transformation*.

Gauss-Weingarten formulas

Let $\mathcal{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be an adapted frame of a surface $p\colon U \to \mathbb{R}^3$. Since $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are perpendicular to ν, there exists a matrix

$$\check{I} = \begin{pmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^2 \end{pmatrix} \qquad \text{such that} \qquad (p_u, p_v) = (\boldsymbol{e}_1, \boldsymbol{e}_2)\,\check{I}\,. \tag{4.29}$$

On the other hand, since $\boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1$, the derivatives of $\boldsymbol{e}_3$ are perpendicular to $\boldsymbol{e}_3$. Then there exists a matrix $\check{I\!I}$ such that

$$\check{I\!I} = \begin{pmatrix} h_1^1 & h_2^1 \\ h_1^2 & h_2^2 \end{pmatrix} \qquad \text{such that} \qquad \big((\boldsymbol{e}_3)_u, (\boldsymbol{e}_3)_v\big) = -(\boldsymbol{e}_1, \boldsymbol{e}_2)\,\check{I\!I}\,. \tag{4.30}$$

Lemma 4.7. *The Gaussian curvature K satisfy*

$$K = \frac{\det \check{I\!I}}{\det \check{I}}$$

Proof. The first and second fundamental matrices are

$$\begin{aligned} \widehat{I} &= \begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (p_u, p_v) = \check{I}^{\,T} \begin{pmatrix} \boldsymbol{e}_1^T \\ \boldsymbol{e}_2^T \end{pmatrix} (\boldsymbol{e}_1, \boldsymbol{e}_2)\,\check{I} = (\check{I}^{\,T})\,\check{I}\,, \\ \widehat{I\!I} &= -\begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (\nu_u, \nu_v) = \check{I}^{\,T} \begin{pmatrix} \boldsymbol{e}_1^T \\ \boldsymbol{e}_2^T \end{pmatrix} (\boldsymbol{e}_1, \boldsymbol{e}_2)\,\check{I\!I} = (\check{I}^{\,T})\,\check{I\!I}\,. \end{aligned}$$

Hence we have the conclusion by (4.11). □

Proposition 4.8. *There exist functions α, β defined on U such that*

$$\mathcal{E}_u = \mathcal{E}\varOmega, \qquad \mathcal{E}_v = \mathcal{E}\varLambda \qquad \left(\varOmega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \varLambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix} \right). \tag{4.31}$$

Proof. Since $\mathcal{E}$ is SO(3)-valued, $\varOmega := \mathcal{E}^{-1}\mathcal{E}_u$ and $\varLambda := \mathcal{E}^{-1}\mathcal{E}_v$ are skew-symmetric matrices. The third columns of $\varOmega$ and $\varLambda$ are nothing but the definition of the matrix $\check{I\!I}$. □

Definition 4.9. The differential form

$$\mu := \alpha\, du + \beta\, dv$$

is called the *connection form* with respect to the adapted frame.

Lemma 4.10. *The connection forms μ and $\tilde{\mu}$ of the adapted frames $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ as in Lemma 4.6 satisfy*

$$\tilde{\mu} = \mu + d\phi.$$

Proof. Let $\widetilde{\varOmega} := \widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{E}}_u$ and $\widetilde{\varLambda} := \widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{E}}_v$. Then

$$\widetilde{\varOmega} = \widetilde{\mathcal{E}}^{-1}(\mathcal{E}_u R + \mathcal{E} R_u) = \widetilde{\mathcal{E}}^{-1}(\mathcal{E}\varOmega R + \mathcal{E} R_u) = \widetilde{\mathcal{E}}^{-1}\widetilde{\mathcal{E}}(R^{-1}\varOmega R + R^{-1}R_u) = R^{-1}\varOmega R + R^{-1}R_u,$$

and $\widetilde{\varLambda} = R^{-1}\varLambda R + R^{-1}R_u$ hold. Then the conclusion follows. □

Exercises

4-1 Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ are given in the form

$$ds^2 = e^{2\sigma}((du^1)^2 + (du^2)^2), \qquad II = \sum_{i,j=1}^{2} h_{ij}\, du^i\, du^j,$$

where σ is a smooth function in (u^1, u^2).

(1) Compute the matrices Ω_j $(j = 1, 2)$ in (4.17).

(2) Set $(u, v) = (u^1, u^2)$, $\boldsymbol{e}_1 := e^{-\sigma} p_{u^1}$, $\boldsymbol{e}_2 := e^{-\sigma} p_{u^2}$, and $\boldsymbol{e}_3 = \nu$, where ν is the unit normal vector field. Compute the matrices Ω and Λ in (4.31) for the orthonormal frame $\mathcal{E} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$.

4-2 Assume the first and second fundamental forms of the surface $p(u^1, u^2)$ are given in the form

$$ds^2 = (du^1)^2 + 2\cos\theta\, du^1\, du^2 + (du^2)^2, \qquad II = 2\sin\theta\, du^1\, du^2,$$

where θ is a smooth function in (u^1, u^2).

(1) Compute the matrices Ω_j $(j = 1, 2)$ in (4.17).

(2) Find an adapted frame, and compute the matrices Ω and Λ in (4.31).

5 The Gauss and Codazzi equations

5.1 Gauss and Codazzi equations

The Gauss-Weingarten formulas (Theorem 4.2) can be considered as a system of partial differential equations with unknown $\mathcal{F}$, whose coefficient matrices are $\varOmega_1$ and $\varOmega_2$.

Remark 5.1. The coefficient matrices $\varOmega_1$ and $\varOmega_2$ in the Gauss-Weingarten formula are expressed in terms of the coefficients of the first and second fundamental forms. In fact, explicit formula for components of $\varOmega_j$ in terms of (g_{ij}) and (h_{ij}) are found in (4.15) and (4.18).

The following proposition is a direct conclusion of Proposition 3.2 and Theorem 4.2:

Proposition 5.2. *Let $p\colon U \to \mathbb{R}^3$ be a parametrized surface defined on a domain U of u^1u^2-plane, and let (g_{ij}) and (h_{ij}) be the coefficients of the first and second fundamental forms. Then the matrices $\varOmega_1$ and $\varOmega_2$ in* (4.17) *satisfy*

$$\frac{\partial \varOmega_1}{\partial u^2} - \frac{\partial \varOmega_2}{\partial u^1} - \varOmega_1\varOmega_2 + \varOmega_2\varOmega_1 = O \tag{5.1}$$

In this section, we show that nine equalities (5.1) are reduced to three equalities, as follows:

Theorem 5.3 (Gauss and Codazzi equations)**.** *The integrability condition* (5.1) *is equivalent to the following three equalities:*

$$h_{11,2} - h_{21,1} = \sum_j \left(\varGamma_{21}^j h_{1j} - \varGamma_{11}^j h_{2j}\right) \tag{5.2}$$

$$h_{12,2} - h_{22,1} = \sum_j \left(\varGamma_{22}^j h_{1j} - \varGamma_{12}^j h_{2j}\right) \tag{5.3}$$

$$K_{ds^2} = \frac{1}{g}(h_{11}h_{22} - h_{12}h_{21})\bigl(= K\bigr), \tag{5.4}$$

where $g := \det(g_{ij}) = g_{11}g_{22} - g_{12}g_{21}$, and

$$K_{ds^2} := \frac{1}{g}R_{12}, \tag{5.5}$$

$$R_{jk} := \frac{1}{2}(g_{1k,2j} - g_{1j,2k} + g_{2j,1k} - g_{2k,1j}) - \sum_{i,s} g_{is}(\varGamma_{k2}^s\varGamma_{1j}^i - \varGamma_{k1}^s\varGamma_{2j}^i) + 2\sum_{l,s} g_{kl}(\varGamma_{s2}^l\varGamma_{1j}^s - \varGamma_{1s}^l\varGamma_{2j}^s). \tag{5.6}$$

The equalities (5.2) and (5.3) are called the *Codazzi equations* , and (5.4) is called the *Gauss equation*.

Remark 5.4. Let

$$h_{ij;k} := h_{ij,k} - \sum_l \left(\varGamma_{ik}^l h_{lj} - \varGamma_{kj}^l h_{il}\right).$$

Then

$$\nabla I\!I := \sum_{i,j,k} h_{ij;k}\, du^i \otimes du^j \otimes du^k$$

does not depend on the coordinate system, which is called the *covariant derivative* of the second fundamental form. The Codazzi equations is equivalent to $h_{ij;k} = h_{ki;j}$, that is, symmetricity of $\nabla I\!I$.

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Remark 5.5. The quantity K_{ds^2} in (5.5) is determined only by the first fundamental form, and one can show that it is invariant under coordinate changes. We call it the (intrinsic) *Gaussian curvature* of ds^2. The Gauss equation (5.4) claims that the intrinsic Gaussian curvature is identical to the Gaussian curvature of the surface.

Proof of Theorem 5.3. We set

$$\begin{pmatrix} I_1^1 & I_2^1 & I_3^1 \\ I_1^2 & I_2^2 & I_3^2 \\ I_1^3 & I_2^3 & I_3^3 \end{pmatrix} := \Omega_{1,2} - \Omega_{2,1} - \Omega_1\Omega_2 + \Omega_2\Omega_1.$$

Then the integrability condition (5.1) is equivalent to $I_j^i = 0$ $(i,j = 1,2,3)$.
Step 1. By symmetricity of h_{ij} and g^{ij},

$$\begin{aligned} I_3^3 &= h_{11}A_2^1 + h_{12}A_2^2 - h_{21}A_1^1 - h_{22}A_1^2 = \sum_l (h_{1l}A_2^l - h_{2l}A_1^l) \\ &= \sum_l \left(h_{1l}\sum_s g^{ls}h_{s2} - h_{2l}\sum_s g^{ls}h_{s1} \right) \\ &= \sum_{l,s} g^{ls}h_{1l}h_{s2} - \sum_{l,s} g^{ls}h_{s1}h_{2l} = \sum_{l,s} g^{ls}h_{1l}h_{s2} - \sum_{l,s} g^{sl}h_{l1}h_{2s} = 0. \end{aligned}$$

Thus the condition $I_3^3 = 0$ is satisfied automatically.
Step 2. Since

$$I_j^3 = h_{1j,2} - h_{2j,1} - \sum_l (\Gamma_{2j}^l h_{l1} - \Gamma_{1j}^l h_{l2}) \qquad (j = 1,2),$$

the conditions $I_j^3 = 0$ $(j = 1,2)$ are equivalent to the Codazzi equations (5.2) and (5.3).
Step 3. For $j = 1,2$

$$\begin{aligned} I_3^j &= -A_{1,2}^j + A_{2,1}^j + \sum_l (\Gamma_{1l}^j A_2^l - \Gamma_{2l}^j A_1^l) \\ &= -\sum_l (g^{jl}h_{1l})_{,2} + \sum_l (g^{jl}h_{l2})_{,1} + \sum_{l,s} g^{ls}(h_{2s}\Gamma_{1l}^j - h_{1s}\Gamma_{2l}^j) \\ &= -\sum_l g^{jl}(h_{1l,2} - h_{l2,1}) - \sum_l (g^{jl}_{,2}h_{1l} - g^{jl}_{,1}h_{l2}) + \sum_{l,s} g^{ls}(h_{2s}\Gamma_{1l}^j - h_{1s}\Gamma_{2l}^j) \\ &= -\sum_l g^{jl}(h_{1l,2} - h_{l2,1}) + \sum_l\sum_{\alpha,\beta} g^{\alpha j}g^{l\beta}(g_{\alpha\beta,2}h_{1l} - g_{\alpha\beta,1}h_{l2}) + \sum_{l,s} g^{ls}(h_{2s}\Gamma_{1l}^j - h_{1s}\Gamma_{2l}^j) \\ &= -\sum_l g^{jl}(h_{1l,2} - h_{l2,1}) + \sum_{l,\alpha,\beta} g^{\alpha j}g^{l\beta}\sum_s \big((g_{\alpha s}\Gamma_{\beta 2}^s + g_{s\beta}\Gamma_{2\alpha}^s)h_{1l} - (g_{\alpha s}\Gamma_{\beta 1}^s + g_{s\beta}\Gamma_{1\alpha}^s)h_{l2}\big) \\ &\qquad + \sum_{l,s} g^{ls}(h_{2s}\Gamma_{1l}^j - h_{1s}\Gamma_{2l}^j) \\ &= -\sum_l g^{jl}(h_{1l,2} - h_{l2,1}) + \sum_{l,\beta} g^{l\beta}\Gamma_{\beta 2}^j h_{1l} + \sum_{l,\alpha} g^{j\alpha}\Gamma_{\alpha 2}^l h_{1l} - \sum_{l,\beta} g^{l\beta}\Gamma_{\beta 1}^j h_{2l} - \sum_{l,\alpha} g^{j\alpha}\Gamma_{\alpha 1}^l h_{2l} \\ &\qquad + \sum_{l,s} g^{ls}(h_{2s}\Gamma_{1l}^j - h_{1s}\Gamma_{2l}^j) \\ &= -\sum_l g^{jl}(h_{1l,2} - h_{l2,1}) - \sum_s (\Gamma_{l2}^s h_{1s} - \Gamma_{1l}^s h_{2s}) = -\sum_l g^{jl} I_l^3, \end{aligned}$$

that is,

$$\begin{pmatrix} I_3^1 \\ I_3^2 \end{pmatrix} = -\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}\begin{pmatrix} I_1^3 \\ I_2^3 \end{pmatrix}.$$

Here, we used Proposition 4.3 and the relation $\widehat{I}^{-1}_{,k} = -\widehat{I}^{-1}\,\widehat{I}_{,k}\,\widehat{I}^{-1}$, i.e.,

$$g^{ij}_{,k} = -\sum_{\alpha\beta} g^{\alpha i} g^{j\beta} g_{\alpha\beta,k}$$

Hence the conditions $I^j_3 = 0$ $(j = 1,2)$ are equivalent to $I^3_j = 0$ $(j = 1,2)$.
Step 4. Since

$$I^i_j = \Gamma^i_{1j,2} - \Gamma^i_{2j,1} - \sum_l (\Gamma^i_{1l}\Gamma^l_{2j} - \Gamma^i_{2l}\Gamma^l_{1j}) + A^i_1 h_{j2} - A^i_2 h_{j1},$$

for $i, j = 1, 2$, we have

$$\sum_i g_{ik} I^i_j = R_{jk} + h_{k1}h_{j2} - h_{k2}h_{j1},$$

where R_{jk} is the quantity given by (5.6). Since the right-most term of the definition of R_{jk} is computed as

$$\sum_{l,s} g_{kl}(\Gamma^s_{1j}\Gamma^l_{s2} - \Gamma^s_{2j}\Gamma^l_{s1}) = \frac{1}{2}\sum_{s,t}\big((g_{k2,s} + g_{sk,2} - g_{2s,k})(g_{tj,1} + g_{1t,j} - g_{1j,t}) \\ - (g_{k1,t} + g_{tk,2} - g_{1k,t})(g_{sj,22} + g_{2s,j} - g_{2j,s})\big),$$

Hence R_{jk} is skew symmetric in j and k:

$$R_{12} = -R_{21}, \qquad R_{11} = R_{22} = 0.$$

Therefore $I^i_j = 0$ for $i, j = 1, 2$ is equivalent to the Gauss equation (5.4). □

5.2 Integrability conditions for orthonormal frames

Under the formulation with orthonormal frame as in Proposition 4.8, we can compute the integrability conditions. Since Ω and Λ are skew-symmetric matrices, the conditions consist of three scalar equalities obviously. Such a formulation will be discussed in the lecture on the next quarter.

Exercises

Let $p\colon U \to \mathbb{R}^3$ be a regular surface of domain $U \subset \mathbb{R}^2$, and denote by $(u^1, u^2) = (u, v)$ the coordinate system of U. And write the first and second fundamental forms as

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = \sum_{i,j} g_{ij}\,du^i\,du^j,$$
$$II = L\,du^2 + 2M\,du\,dv + N\,dv^2 = \sum_{i,j} h_{ij}\,du^i\,du^j,$$

respectively.

5-1 Assume $L = N = 0$, that is, $II = 2M\,du\,dv = 2h_{12}\,du^1\,du^2$, Prove that, if the Gaussian curvature K is negative constant,

$$E_v = G_u = 0, \qquad \text{that is,} \qquad g_{11,2} = g_{22,1} = 0.$$

5-2 Assume $F = 0$ and $E = G = e^{2\sigma}$, where σ is a function in (u, v). Let $z = u + iv$ ($i = \sqrt{-1}$) and define a complex-valued function q in z by

$$q(z) := \frac{L(u,v) - N(u,v)}{2} - iM(u,v).$$

Prove that the Codazzi equations are equivalent to

$$\frac{\partial q}{\partial \bar{z}} = e^{2\sigma}\frac{\partial H}{\partial z},$$

where H is the mean curvature, and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

6 The fundamental theorem for surfaces

6.1 The statement

Let U be a domain of u^1u^2-plane and let

$$\widehat{I} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad \widehat{II} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \tag{6.1}$$

be two symmetric matrices whose components are real-valued C^∞-functions on U. In addition, assume

$$g_{11} > 0, \qquad g_{22} > 0, \quad \text{and} \qquad g_{11}g_{22} - g_{12}g_{21} > 0 \tag{6.2}$$

hold on U. In other words, $\widehat{I}$ is a positive-definite matrix at each point on U. Define

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{2} g^{kl}(g_{kj,i} + g_{ik,j} - g_{ij,k}), \qquad A^i_j = \sum_{l=1}^{2} g^{il}h_{lj} \tag{6.3}$$

where $(g^{ij}) = (g_{ij})^{-1}$ is the inverse matrix of (g_{ij}).

Theorem 6.1 (The fundamental theorem for surface theory)**.** *Assume U is simply connected, and (g_{ij}) and (h_{ij}) satisfy the Gauss equation* (5.4) *and the Codazzi equations* (5.2)–(5.3) *in the previous section. Then there exists a regular surface $p\colon U \to \mathbb{R}^3$ such that*

- *the first fundamental form of p is $ds^2 = \sum_{i,j} g_{ij}du^i\,du^j$,*
- *the second fundamental form of p with respect to the unit normal vector field $\nu := (p_{,1} \times p_{,2})/|p_{,1} \times p_{,2}|$ coincides with $II = \sum_{i,j} h_{ij}du^i\,du^j$.*

Moreover, such a surface p is unique up to a transformation

$$p \mapsto Rp + \boldsymbol{a}, \qquad R \in \mathrm{SO}(3),\ \boldsymbol{a} \in \mathbb{R}^3.$$

6.2 Uniqueness

Here we shall prove the uniqueness part of Theorem 6.1. Let p and $\tilde{p}$ be regular surfaces in $\mathbb{R}^3$ defined on a domain U of u^1u^2-plane[10], with unit normal vector fields

$$\nu := \frac{p_{,1} \times p_{,2}}{|p_{,1} \times p_{,2}|} \qquad \text{and} \qquad \tilde{\nu} := \frac{\tilde{p}_{,1} \times \tilde{p}_{,2}}{|\tilde{p}_{,1} \times \tilde{p}_{,2}|},$$

respectively. Then the Gauss frame of p and $\tilde{p}$ are written as

$$\mathcal{F} := (p_{,1}, p_{,2}, \nu), \qquad \text{and} \qquad \widetilde{\mathcal{F}} := (\tilde{p}_{,1}, \tilde{p}_{,2}, \tilde{\nu}),$$

respectively. Assume the coefficients (g_{ij}) and $(h_{ij})$of the first and second fundamental forms are common for p and $\tilde{p}$. Then $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ satisfy the Gauss-Weingarten equations (4.17)

$$\mathcal{F}_{,j} = \mathcal{F}\Omega_j \quad \text{and} \quad \widetilde{\mathcal{F}}_{,j} = \widetilde{\mathcal{F}}\Omega_j, \qquad \text{where} \quad \Omega_j = \begin{pmatrix} \Gamma^1_{1j} & \Gamma^1_{2j} & -A^1_j \\ \Gamma^2_{1j} & \Gamma^2_{2j} & -A^2_j \\ h_{1j} & h_{2j} & 0 \end{pmatrix}. \tag{6.4}$$

Hence, for $i = 1, 2$,

$$\frac{\partial}{\partial u^j}\widetilde{\mathcal{F}}\mathcal{F}^{-1} = \widetilde{\mathcal{F}}_{,j}\mathcal{F}^{-1} + \widetilde{\mathcal{F}}(\mathcal{F}^{-1})_{,j} = \widetilde{\mathcal{F}}_{,j}\mathcal{F}^{-1} - \widetilde{\mathcal{F}}\mathcal{F}^{-1}\mathcal{F}_{,j}\mathcal{F}^{-1} = \widetilde{\mathcal{F}}\Omega_j\mathcal{F}^{-1} - \widetilde{\mathcal{F}}\Omega_j\mathcal{F}^{-1} = O$$

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[10]The uniqueness does not require simple connectedness of U.

hold on U. Since we have assumed that U is a domain, U is (arcwise) connected. This implies that $R := \widetilde{\mathcal{F}}\mathcal{F}^{-1}$ is a constant matrix on U. Moreover, since p and $\tilde{p}$ share their first fundamental forms, it holds that

$$\mathcal{F}^T\mathcal{F} = \begin{pmatrix} p_{,1}\cdot p_{,1} & p_{,1}\cdot p_{,2} & p_{,1}\cdot\nu \\ p_{,2}\cdot p_{,1} & p_{,2}\cdot p_{,2} & p_{,2}\cdot\nu \\ \nu\cdot p_{,1} & \nu\cdot p_{,2} & \nu\cdot\nu \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \widetilde{\mathcal{F}}^T\widetilde{\mathcal{F}} = \mathcal{F}^T R^T R\mathcal{F}.$$

Hence $R^TR = \mathrm{id}$, that is, R is an orthogonal matrix. Moreover,

$$\tilde{\nu} = \frac{\tilde{p}_{,1}\times\tilde{p}_{,2}}{|\tilde{p}_{,1}\times\tilde{p}_{,2}|} = R\nu = R\frac{p_{,1}\times p_{,2}}{|p_{,1}\times p_{,2}|}$$

implies $R(p_{,1}\times p_{,2}) = (Rp_{,1})\times(Rp_{,2})$, hence $\det R = 1$. Summing up, the Gauss frames $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are related as $\widetilde{\mathcal{F}} = R\mathcal{F}$ ($R\in\mathrm{SO}(3)$). By the first and second columns of this relation, it holds that

$$d\tilde{p} = \tilde{p}_{,1}\,du^1 + \tilde{p}_{,2}\,du^2 = Rp_{,1}du^1 + Rp_{,2}\,du^2 = R(dp).$$

Hence, by connectivitity of U again, $\boldsymbol{a} := \tilde{p} - Rp$ is a constant vector. □

6.3 Existence

Next, we show the existence part of Theorem 6.1.

Lemma 6.2. *Let (γ_{ij}) be a positive definite symmetric matrix, that is, γ_{11} and γ_{22} are positive, $\gamma_{11}\gamma_{22}-\gamma_{12}\gamma_{21} > 0$ and $\gamma_{12}=\gamma_{21}$. Then there exists a vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ in $\mathbb{R}^3$ such that*

$$\boldsymbol{v}_i\cdot\boldsymbol{v}_j = \gamma_{ij},\qquad \boldsymbol{v}_3\cdot\boldsymbol{v}_j = 0,\qquad \boldsymbol{v}_3\cdot\boldsymbol{v}_3 = 1,\qquad \textit{and}\qquad \det(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3) > 0$$

hold for i, $j = 1, 2$.

Proof. Let $\theta\in(0,\pi)$ be an angle satisfying $\cos\theta = \gamma_{12}/\sqrt{\gamma_{11}\gamma_{22}} \in (-1,1)\setminus\{0\}$, and set

$$\boldsymbol{v}_1 := \sqrt{\gamma_{11}}\begin{pmatrix}1\\0\\0\end{pmatrix},\qquad \boldsymbol{v}_2 := \sqrt{\gamma_{22}}\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix},\qquad \boldsymbol{v}_3 := \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Then $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are desired vectors. □

Step 1. We fix a point P_0 in U. Then by Lemma 6.2, there exists a matrix $\mathcal{F}_0$ such that

$$\mathcal{F}_0^T\mathcal{F}_0 = \begin{pmatrix} g_{11}(\mathrm{P}_0) & g_{12}(\mathrm{P}_0) & 0 \\ g_{21}(\mathrm{P}_0) & g_{22}(\mathrm{P}_0) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.5}$$

Since (g_{ij}) and (h_{ij}) satisfy the Gauss and Codazzi equations, Theorem 5.3 implies that the equation (6.4) for unknown matrix-valued function $\mathcal{F}$. So, by Theorem 3.5, there exists a unique matrix-valued function $\mathcal{F}$ defined on U satisfying

$$\mathcal{F}_{,j} = \mathcal{F}\Omega_j,\qquad \mathcal{F}(\mathrm{P}_0) = \mathcal{F}_0 \tag{6.6}$$

for a matrix $\mathcal{F}_0$ satisfying (6.5). Decompose the solution $\mathcal{F}$ into column vectors as

$$\mathcal{F}(u^1,u^2) = (\boldsymbol{a}_1(u^1,u^2), \boldsymbol{a}_2(u^1,u^2), \boldsymbol{a}_3(u^1,u^2)).$$

Then it holds that

$$\frac{\partial}{\partial u^2}(\boldsymbol{a}_1) = \mathit{\Gamma}_{12}^1\boldsymbol{a}_1 + \mathit{\Gamma}_{12}^2\boldsymbol{a}_2 + h_{12}\boldsymbol{a}_3,$$
$$\frac{\partial}{\partial u^1}(\boldsymbol{a}_2) = \mathit{\Gamma}_{21}^1\boldsymbol{a}_1 + \mathit{\Gamma}_{21}^2\boldsymbol{a}_2 + h_{21}\boldsymbol{a}_3,$$

that is,

$$\omega := \boldsymbol{a}_1\,du^1 + \boldsymbol{a}_2\,du^2$$

is a (vector-valued) closed one-form on the simply connected domain U. Hence by Poincaré's lemma (Theorem 1.9), there exists a map $p\colon U \to \mathbb{R}^3$ with $dp = \omega$, that is,

$$p_{,1} = \boldsymbol{a}_1, \qquad p_{,2} = \boldsymbol{a}_2. \tag{6.7}$$

<u>*Step 2.*</u> We shall show that p obtained in the previous step is the desired one. Let $\mathcal{F}$ be a solution of (6.6). Then the symmetric matrix-valued function $\mathcal{F}^T\mathcal{F}$ satisfies a system of linear partial differential equations

$$\frac{\partial \mathcal{F}^T\mathcal{F}}{\partial u^j} = \mathit{\Omega}_j^T\mathcal{F}^T\mathcal{F} + \mathcal{F}^T\mathcal{F}\mathit{\Omega}_j, \qquad \mathcal{F}^T\mathcal{F}(\mathrm{P}_0) = \mathcal{F}_0^T\mathcal{F}_0$$

where $\mathcal{F}_0$ is a matrix as in (6.5).

On the other hand, consider the matrix-valued function

$$\mathcal{G} := \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, by (6.3), it holds that

$$\mathcal{G}_{,j} = \mathit{\Omega}_j^T\mathcal{G} + \mathcal{G}\mathit{\Omega}_j \qquad \mathcal{G}(\mathrm{P}_0) = \mathcal{F}_0^T\mathcal{F}_0. \tag{6.8}$$

Hence $\mathcal{F}^T\mathcal{F}$ and $\mathcal{G}$ satisfy the same system of partial differential equations with the same initial conditions. Thus, the uniqueness of the solution infers $\mathcal{F}^T\mathcal{F} = \mathcal{G}$, that is,

$$\begin{pmatrix} \boldsymbol{a}_1\cdot\boldsymbol{a}_1 & \boldsymbol{a}_1\cdot\boldsymbol{a}_2 & \boldsymbol{a}_1\cdot\boldsymbol{a}_3 \\ \boldsymbol{a}_2\cdot\boldsymbol{a}_1 & \boldsymbol{a}_2\cdot\boldsymbol{a}_2 & \boldsymbol{a}_2\cdot\boldsymbol{a}_3 \\ \boldsymbol{a}_3\cdot\boldsymbol{a}_1 & \boldsymbol{a}_3\cdot\boldsymbol{a}_2 & \boldsymbol{a}_3\cdot\boldsymbol{a}_3 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So, together with (6.7) and that $\det\mathcal{F} > 0$

$$g_{ij} = p_{,i}\cdot p_{,j}, \qquad \nu = \boldsymbol{a}_3.$$

Then

$$h_{ij} = (\boldsymbol{a}_i)_{,j}\cdot\nu = p_{,ij}\cdot\nu,$$

that is, the coefficients of the second fundamental form coincides with (h_{ij}). $\square$

Exercises

6-1 Let $\theta\colon U \to \mathbb{R}$ be a C^∞-function defined on a simply connected domain U of the uv-plane $\mathbb{R}^2$. Assuming θ satisfies $\theta_{uv} = \sin\theta$, prove that there exists a surface $p\colon U \to \mathbb{R}^3$ whose first and second fundamental forms are

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2, \qquad I\!I = 2\sin\theta\, du\, dv.$$

6-2 Let $\sigma\colon U \to \mathbb{R}$ be a C^∞-function defined on a simply connected domain U of the uv-plane $\mathbb{R}^2$. Assuming σ satisfies $\Delta\sigma = -\frac{1}{2}\sinh 2\sigma$, prove that there exists a surface $p\colon U \to \mathbb{R}^3$ with

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad I\!I = \frac{1}{2}\big((e^{2\sigma}+1)du^2 + (e^{2\sigma}-1)dv^2\big).$$

7 An application—Surfaces of constant mean curvature

7.1 Mean curvature

Let $p\colon U \ni (u,v) \mapsto p(u,v) \in \mathbb{R}^3$ be a regular parametrization of a surface defined on a domain $U \subset \mathbb{R}^2$, and let ν be its unit normal vector field. We write first and second fundamental forms as

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad I\!I = L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

where

$$(\widehat{I} :=) \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} p_u \cdot p_u & p_u \cdot p_v \\ p_v \cdot p_u & p_v \cdot p_v \end{pmatrix}, \qquad (\widehat{I\!I} :=) \begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} \nu_u \cdot p_u & \nu_u \cdot p_v \\ \nu_v \cdot p_u & \nu_v \cdot p_v \end{pmatrix}.$$

Since the parametrization is regular, the matrix $\widehat{I}$ is positive definite:

$$EG - F^2 > 0, \quad E > 0, \quad G > 0.$$

Then we define the Weingarten matrix A by

$$A = \widehat{I}^{-1}\,\widehat{I\!I}\,.$$

Definition 7.1. The *mean curvature* of the surface p is defined by

$$H := \frac{1}{2}\operatorname{tr} A = \frac{EN - 2FM + GL}{2(EG - F)^2}.$$

7.2 Area and mean curvature

To explain geometric meanings of mean curvature, we start with the area of surfaces: Let $p\colon U \to \mathbb{R}^3$ be a regular parametrization of a surface as in the top of this subsection. Take a subdomain $V \subset U$ such that the closure $\overline{V}$ of V is bounded and contained in U.

Definition 7.2. The *area* of the image $p(\overline{V})$ of the surface is defined as

$$\mathcal{A}_p(\overline{V}) := \iint_{\overline{V}} da, \qquad da := \sqrt{\det \widehat{I}}\,du\,dv = \sqrt{EG - F^2}\,du\,dv.$$

We call da the *area element* of p.

For a real number t, $p^t := p + t\nu$ is called the *parallel surface* of p with distance t.

Proposition 7.3.

$$\mathcal{A}_{p^t}(\overline{V}) = \mathcal{A}_p(\overline{V}) - 2t \iint_{\overline{V}} H\,da + o(t) \qquad (t \to 0).$$

Proof. The coefficient matrix of the first fundamental form of p^t is obtained as

$$\begin{aligned} \widehat{I}^{\,t} := \begin{pmatrix} E^t & F^t \\ F^t & G^t \end{pmatrix} &= \begin{pmatrix} (p_u + t\nu_u)\cdot(p_u + t\nu_u) & (p_u + t\nu_u)\cdot(p_v + t\nu_v) \\ (p_v + t\nu_v)\cdot(p_u + t\nu_u) & (p_v + t\nu_v)\cdot(p_v + t\nu_v) \end{pmatrix} \\ &= \begin{pmatrix} E - 2tL & F - 2tM \\ F - 2tM & G - 2tN \end{pmatrix} + o(t) = \widehat{I} - 2t\,\widehat{I\!I} + o(t). \end{aligned}$$

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Then

$$\det \widehat{I}^{\,t} = (EG - F^2) - 2t(EN - 2FM + GL) + o(t)$$
$$= (EG - F^2)\left(1 - 2t\frac{EN - 2FM + GL}{EG - F^2} + o(t)\right) = (EG - F^2)\,(1 - 4tH + o(t))\,.$$

Hence the area element of p^t is

$$da^t := \sqrt{\det\,\widehat{I}}\,du\,dv = \sqrt{EG - F^2}\sqrt{1 - 4tH + o(t)}\,du\,dv = \bigl(1 - 2tH + o(t)\bigr)\,da$$

Integrating this, we obtain the conclusion. □

Roughly speaking, the mean curvature is the rate of change of the area of a family of parallel surfaces of a surface. The following proposition supports this: We denote by $\overline{D}$ and $S^1 = \partial D$ the unit closed disc $\{(u,v)\,;\,u^2 + v^2 \leqq 1\}$ and its boundary, respectively. Let $C \subset \mathbb{R}^3$ be a simple closed curve in $\mathbb{R}^3$ and denote $\mathcal{S}_C$ the set of surfaces $p\colon \overline{D} \to \mathbb{R}^3$ with $p(S^1) = C$.

Fact 7.4. *If a surface $p \in \mathcal{S}_C$ has the least area among all surfaces in $\mathcal{S}_C$, then the mean curvature of p identically vanishes.*

If you are familiar to the variational method, this means that the Euler-Lagrange equation of the area functional $\mathcal{A}\colon \mathcal{S}_C \to \mathbb{R}$ is "$H = 0$". Keeping this fact in mind,

Definition 7.5. A *minimal surface* is a surface whose mean curvature vanishes identically.

On the other hand, the conditional extremal problem for the area functional, we have

Fact 7.6. *When the volume of the enclosed domain is fixed, the closed surface with the least area is of* (*non-zero*) *constant mean curvature.*

7.3 Examples of constant mean curvature surfaces

Since the mean curvature is invariant under congruence of $\mathbb{R}^3$, we have

Lemma 7.7. *Let $S \subset \mathbb{R}^3$ be a surface* (*an image of a parametrized surface*)*. Assume for all* P *and* $\mathrm{Q} \in S$*, there exists an orientation preserving congruence F of the Euclidean* 3*-space satisfying $F(S) = S$ and $F(\mathrm{P}) = \mathrm{Q}$. Then the mean curvature of S is constant.*

Example 7.8 (The plane)**.** A plane $p(u,v) = (u,v,0)$ is a minimal surface. In fact, since the unit normal vector field $\nu = (0,0,1)$ is constant, $I\!I$ vanishes identically.

Example 7.9 (The round sphere)**.** Let $S := S^2(r) \subset \mathbb{R}^3$ be the sphere of radius $r > 0$ centered at the origin. Since the linear action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$ preserves $S^2(r)$ and transitive, the mean curvature of $S^2(r)$ is constant.

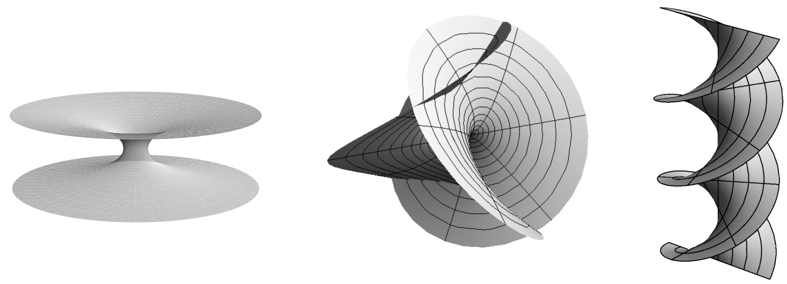
Let us compute the value of the mean curvature: For each point $\boldsymbol{p} \in S^2(r)$, the position vector $\boldsymbol{p}$ is perpendicular to the tangent plane of $S^2(r)$ at $\boldsymbol{p}$. Hence $\boldsymbol{\nu} := (1/r)\boldsymbol{p}$ is the (outward) unit normal vector.

Consider the parallel surface

$$S^t := \left\{\boldsymbol{p} + t\boldsymbol{\nu} = \left(1 + \frac{t}{r}\right)\boldsymbol{p}\,;\,\boldsymbol{p} \in S = S^2(r)\right\},$$

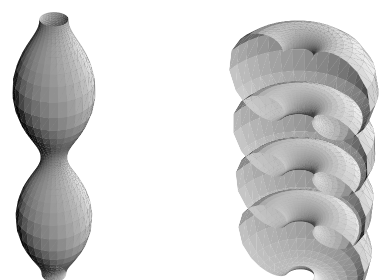
which is the sphere ov radius $(1 + t/r)$. Then

$$\text{Area of } S^t - \text{Area of } S = 4\pi\,(r+t)^2 - 4\pi r^2 = 8\pi rt + O(t^2).$$



catenoid Enneper's helicoid

Figure 1: Minimal surfaces (cf. [UY17])



unduloid nodoid

Figure 2: Delaunay's surfaces (constant mean curvature) (cf. [UY17])

Since the mean curvature H is constant, Proposition 7.3 yields that

$$8\pi r t = -2t \iint_S H\,dA = -2tH(\text{area of } S) = -t \times 8\pi r^2 H.$$

Hence the mean curvature (with respect to the outward unit normal) is $-1/r$.

Similarly, the mean curvature with respect to the inward unit normal is $1/r$.

Example 7.10 (The cylinder)**.** Let S be a circular cylinder of radius r whose axis is the vertical axis of $\mathbb{R}^3$:

$$S = \{\boldsymbol{x} = (x, y, z)\,;\, x^2 + y^2 = r^2\}.$$

Since rotations around the z-axis and vertical translations acts on S transitively, the mean curvature is constant. The same argument as in Example 7.9 works for a finite strip $S' := \{(x, y, z) \in S\,;\, 0 \leqq z \leqq 1\}$, for example, and one can deduce the mean curvature with respect to outward (resp. inward) unit normal is $-1/(2r)$ (resp. $1/2r$).

Question 7.11. *Are there any other constant mean curvature surfaces than the "trivial" examples above?*

7.4 Constant mean curvature surfaces

There are number of examples of constant mean curvature, see Figures 1 and 2.

On the other hand, the following uniqueness theorems are obtained in the middle of 20th century. Here, a *closed surface* means an immersion $p\colon S \to \mathbb{R}^3$ of a compact 2-manifold without boundary.

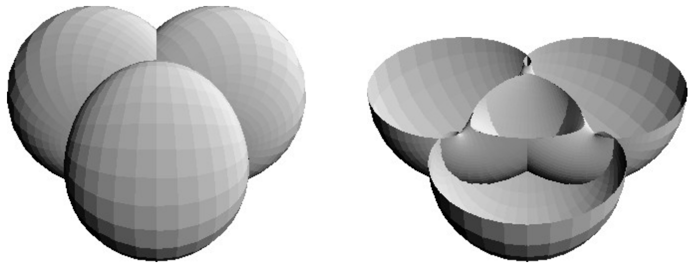


Figure 3: Wente torus

Fact 7.12 (A. D. Alexandrov[Ale58]). *The only closed surfaces of constant mean curvature without self-intersections are the round spheres.*

Fact 7.13 (H. Hopf [Hop53]). *The only closed surfaces of constant mean curvature whose genus zero are the round spheres.*

Then the following problem arises:

Question 7.14 (Hopf's problem). *Are there closed surfaces of constant mean curvature other than the round spheres.*

In 1986, H. Wente constructed constant mean curvature torus [Wen86a] (see Figure 3). Besides, N. Kapouleas also gave examples of constant mean curvature surfaces of genus $\geqq 2$ [Wen86b, BK14]. These two results are obtained by quite different methods. In this lecture, an outline of Wente's construction is introduced as an application of the fundamental theorem for surface theory.

7.5 Wente torus

In this section, we outline the construction of constant mean curvature tori according to Wente [Wen86a].

Definition 7.15. A function f defined on $\mathbb{R}^2$ is said to be *doubly periodic* if there exists a pair $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ of linearly independent vectors in $\mathbb{R}^2$ such that

$$f(\boldsymbol{x}+\boldsymbol{v}_1) = f(\boldsymbol{x}+\boldsymbol{v}_2) = f(\boldsymbol{x})y \tag{7.1}$$

holds for any $\boldsymbol{x} \in \mathbb{R}^2$. The basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is called the *period* of f. n

Remark 7.16. If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is doubly periodic with period $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$,

$$f(\boldsymbol{x} + m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2) = f(\boldsymbol{x}) \qquad \boldsymbol{x} \in \mathbb{R}^2$$

holds for all $(m_1, m_2) \in \mathbb{Z}^2$. In other words, the function f is invariant under the action of the abelian group

$$\varGamma := \mathbb{Z}\boldsymbol{v}_1 \oplus \mathbb{Z}\boldsymbol{v}_2$$

on $\mathbb{R}^2$ as translations.

Since the quotient space $T := \mathbb{R}^2/\varGamma$ is a smooth 2-manifold diffeomorphic to the torus, the doubly periodic function f is considered as a function on T .

So our goal is

- to construct a doubly periodic constant mean curvature immersion $p\colon \mathbb{R}^2 \to \mathbb{R}^3$.

For the construction, we apply the fundamental theorem for surface theory:

Proposition 7.17. *Let $\sigma\colon \mathbb{R}^2 \to \mathbb{R}$ be a doubly periodic function with period $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$. If σ satisfies*

$$\Delta\sigma = \sigma_{uu} + \sigma_{vv} = -\frac{1}{2}\sinh 2\sigma, \tag{7.2}$$

there exists a parametrized surface $\boldsymbol{p}\colon \mathbb{R}^2 \to \mathbb{R}^3$ with

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad II = \frac{1}{2}\bigl((e^{2\sigma}+1)du^2 + (e^{2\sigma}-1)dv^2\bigr), \tag{7.3}$$

whose mean curvature is identically $1/2$. Moreover, there exist matrices $R_i \in \mathrm{SO}(3)$ and vectors $\boldsymbol{a}_i \in \mathbb{R}^3$ $(i = 1, 2)$ such that

$$\boldsymbol{p}(\boldsymbol{x} + \boldsymbol{v}_i) = R_i\boldsymbol{p}(\boldsymbol{x}) + \boldsymbol{a}_i \qquad (i = 1, 2) \tag{7.4}$$

holds for all $\boldsymbol{x} \in \mathbb{R}^2$.

Proof. Exercise 6-2 yields the existence of $\boldsymbol{p}$ with (7.3). Moreover, since $\sigma(\boldsymbol{x} + \boldsymbol{v}_i) = \sigma(\boldsymbol{x})$, $\boldsymbol{p}(\boldsymbol{x} + \boldsymbol{v}_i)$ and $\boldsymbol{p}(\boldsymbol{x})$ have common first and second fundamental forms. Hence the uniqueness of the fundamental theorem implies the existence of R_i and $\boldsymbol{a}_i$ as (7.4). □

In [Wen86a, Section IV], Wente constructed the solutions of (7.2) as follows:

> Let a and b be positive numbers, and set $\Omega = [0, a] \times [0, b] \subset \mathbb{R}^2$, which is a closed rectangle. First, consider the boundary value problem
>
> $$\Delta\sigma = -\frac{1}{2}\sinh 2\sigma \quad \text{on } \Omega, \qquad \sigma = 0 \quad \text{on } \partial\Omega, \qquad \sigma > 0 \quad \text{on } \Omega^o,$$
>
> where Ω^o is the interior of Ω. Then by reflecting this solution about boundaries, one can extend σ on whole $\mathbb{R}^2$, and the resulting function is doubly periodic with period $\{(2a, 0), (0, 2b)\}$.

Observing the symmetries of σ, one can deduce that $R_2 = \mathrm{id}$, $\boldsymbol{a}_i = \boldsymbol{0}$ $(i = 1, 2)$, and

$$R_1 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\theta = \theta(a, b)$ is a real number. Moreover, one can show that θ is a non-constant continuous function in (a, b). Hence there exists (a, b) such that $\theta = \theta(a, b) \in 2\pi\mathbb{Q}$. For such (a, b), $R_1^m = \mathrm{id}$ for some integer m. This means that $\boldsymbol{p}$ is $\{(ma, 0), (0, b)\}$-periodic, which yields the example.

After Wente, a lot of results related Wente-type tori are obtained. See, for example, [Abr87, Wal87, PS89].

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Glossary

1-form 微分 1-形式, 3

adapted frame 適合枠, 23
arc-length parameter 弧長径数, 11
area element 面積要素, 35
area functional 面積汎関数, 36
area 面積, 35

Cauchy-Riemann equations コーシー・リーマン方程式, 16
Christoffel symbols クリストッフェル記号, 21
Codazzi equations コダッチ方程式, 27
column vector 列ベクトル, 1, 7
commutativity 可換性, 2
compatibility condition 適合条件, 13
conjugate 共役, 17
connection form 接続形式, 25
covariant derivative 共変微分, 27
curvature 曲率, 11

determinant 行列式, 1
differential 2-form 微分 2-形式, 3
differential form 微分形式, 3
differential one form 微分 1-形式, 3
differential 微分, 3
distance 距離, 1, 19
domain 領域, 1

eigenvalue 固有値, 7
Euclidean space ユークリッド空間, 1, 19
Euler-Lagrange equation オイラー・ラグランジュ方程式, 36
exterior differential 外微分, 3

Frenet frame フルネ枠, 11

gauge transformation ゲージ変換, 24
Gauss frame ガウス枠, 21
Gauss-Weingarten formula ガウス・ワインガルテンの公式, 21
Gaussian curvature ガウス曲率, 20
general linear group ($\mathrm{GL}(n,\mathbb{R})$) 一般線形群, 7

harmonic function 調和関数, 16

holomorphic 正則（複素関数が）, 16

identity matrix 単位行列, 1
immersion はめ込み, 19
initial value problem 初期値問題, 5
inner product 内積, 1, 19
integrability condition 可積分条件, 13
isometry 等長写像，等長変換, 1
isometry 等長変換, 19

Kronecker's delta symbol クロネッカーのデルタ記号, 21

Laplacian ラプラシアン, 16
latitude 緯度, 2
linear function 1 次関数, 6
linear ordinary differential equation 線形常微分方程式, 6
longitude 経度, 2

map 写像, 1
matrix 行列, 1
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Mercator's world map メルカトルの世界地図, 2
minimal surface 極小曲面, 36

norm ノルム, 1, 7, 19

ODE → ordinary differential equation, 5
ordinary differential equation 常微分方程式, 5
orientation preserving 向きを保つ, 1
origin 原点, 36
orthogonal group ($\mathrm{O}(n)$) 直交群, 8
orthogonal matrix 直交行列, 1
outer product 外積, 19

parallel surface 平行曲面, 35
parametrization パラメータ表示, 2, 19
partial derivative 偏微分，偏導関数, 2
partial differential equation 偏微分方程式, 13
perpendicular 垂直, 36
plane 平面, 36
Poincaré lemma ポアンカレの補題, 4