Advanced Topics in Geometry B1 (MTH.B406)

Surfaces of constant Gaussian curvatrure

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Exercise 1-1

Show Lemma 1.12:

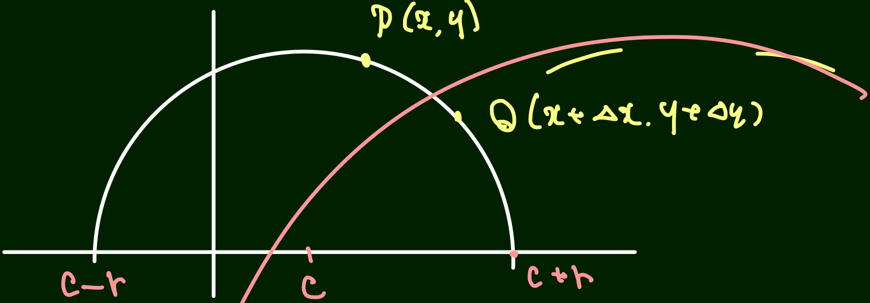
Lemma

Let $C_{c,r}$ be a circle passing through $\mathrm{P} = (x, y)$ and $\mathrm{Q} = (x + \Delta x, y + \Delta y)$, where $\Delta x \neq 0$, and set

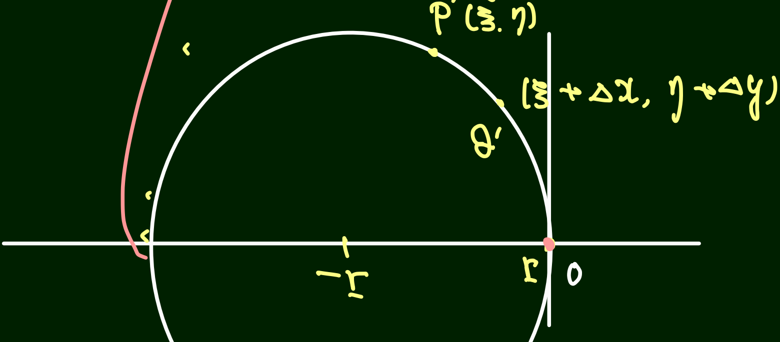
$$(X, Y) := \psi_{2r} \circ \tau_{-c-r}(x, y),$$
$$(X + \Delta X, Y + \Delta Y) := \psi_{2r} \circ \tau_{-c-r}(x + \Delta x, y + \Delta y).$$

Then $\Delta X = 0$ and

$$\frac{\Delta Y}{Y} = \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{y} + o\left(\sqrt{(\Delta x)^2 + (\Delta y)^2}\right).$$



$$T_{-c-r}(x, y) = (x - c - r,\ y) =: (\xi, \eta)$$



Exercise 1-1

$$(\xi, \eta) := \tau_{-c-r}(x, y) = (x - c - r, y)$$
$$\Rightarrow \qquad \tau_{-c-r}(x + \Delta x, y + \Delta y) = (\xi + \Delta x, \eta + \Delta y)$$

$(\xi + r)^2 + \eta^2 = r^2$

- Both (ξ, η), $(\xi + \Delta x, \eta + \Delta y)$ lie on $C_{-r,r}$.

$$\xi^2 + \eta^2 = -2r\xi, \qquad (\xi + \Delta x)^2 + (\eta + \Delta y)^2 = -2r(\xi + \Delta x).$$

$$\psi_{2r}(\xi, \eta) = \frac{4r^2}{\xi^2 + \eta^2}(\xi, \eta) = -(2r)^{-1}\left(1, \frac{\eta}{\xi}\right) = (X, Y)$$

$\boxed{\Delta X = 0}$

$$\psi_{2r}(\xi + \Delta x, \eta + \Delta y) = -(2r)^{-1}\left(1, \frac{\eta + \Delta y}{\xi + \Delta x}\right) = (X + \Delta X, Y + \Delta Y).$$

Exercise 1-1

- $(\xi + \Delta x, \eta + \Delta y)$ lies on $C_{-r,r}$:

$$\underline{(\xi + \Delta x)^2 + (\eta + \Delta y)^2} = \underline{-2r(\xi + \Delta x)}$$

$$\Rightarrow \quad \underline{(\xi + r)\Delta x + \eta \Delta y} = o(\sqrt{(\Delta x)^2 + (\Delta y)^2})$$

- (ξ, η) lies on $C_{-r,r}$:

$$r^2 = (\xi + r)^2 + \eta^2 = \left(\eta \frac{\Delta y}{\Delta x}\right)^2 + \eta^2 + o(1)$$

$$= \underline{\eta^2 \left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)} + o(1)$$

$$r = \eta \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

Exercise 1-1

$$\frac{1}{\left(1+\frac{\Delta x}{\xi}\right)} \approx 1-\frac{\Delta x}{\xi}+o(\Delta x)$$

$$\begin{aligned}
\frac{\Delta Y}{Y} &= \cancel{-2r}\left(\frac{\eta+\Delta y}{\xi+\Delta x}-\frac{\eta}{\xi}\right)\Big/\left(\cancel{-2r}\frac{\eta}{\xi}\right)\\
&= \frac{\xi}{\eta}\left(\frac{1}{\xi}(\eta+\Delta y)\left(1-\frac{\Delta x}{\xi}\right)-\frac{\eta}{\xi}\right)+o(*)\\
&= \frac{\Delta y}{\eta}-\frac{\Delta x}{\xi}+o(*)\\
&= -\left(\frac{\xi+r}{\eta^2}+\frac{1}{\xi}\right)\Delta x+o(*)\\
&= -\frac{\xi(\xi+r)+\eta^2}{\xi\eta^2}+o(*)\\
&= \frac{1}{\eta}\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}+o(*).
\end{aligned}$$

$$* = \sqrt{(\Delta x)^2+(\Delta y)^2}$$

$$\Delta y = -\frac{\xi+r}{\eta}\Delta x$$

$$(\xi+r)^2+\eta^2 \quad r^2$$

$$-r\xi$$

$\underline{y = \eta}$

$$k \frac{\Delta Y}{Y} = \frac{k}{y} \sqrt{(\Delta x)^2 + (\Delta y)^2} + o(*)$$

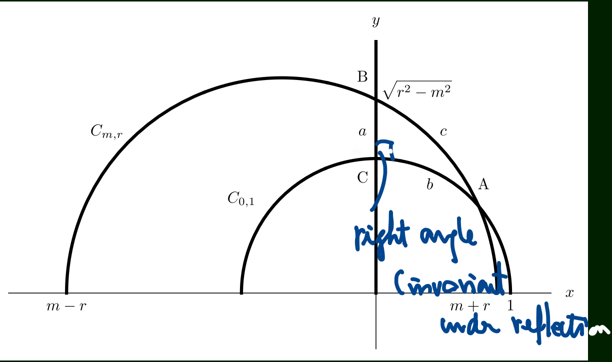
distance of (X, Y) & $(X + \Delta X, Y + \Delta Y)$

dist of P, Q

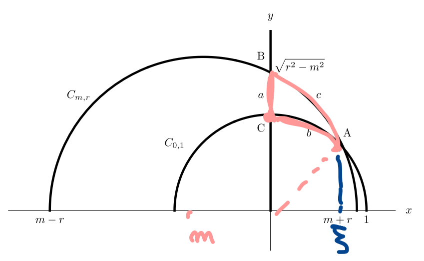
Exercise 1-2

Problem (Ex. 1-2)

Let A *be the intersection point of two lines* $C_{0,1}$ *and* $C_{m,r}$ $(r > 0$, $r^2 - m^2 > 1$, $r + m < 1)$, *and set* $\mathrm{B} = (0, \sqrt{r^2 - m^2})$, *and* $\mathrm{C} = (0, 1)$. *Find a relation of* $a := \mathrm{dist}(\mathrm{B}, \mathrm{C})$, $b := \mathrm{dist}(\mathrm{C}, \mathrm{A})$ *and* $c := \mathrm{dist}(\mathrm{A}, \mathrm{B})$.



Exercise 1-2



$\mathrm{A} = (\xi, \eta);\ \xi = \left(\frac{1+m^2-r^2}{2m}\right)$

$$\mathrm{CB}: (0, t) \qquad t \in (1, \sqrt{r^2 - m^2})$$

$$\mathrm{AC}: (\cos t, \sin t) \qquad t \in (t_1, \tfrac{\pi}{2}), \qquad t_1 = \cos^{-1}\xi,$$

$$\mathrm{AB}: (r\cos t + m, r\sin t) \qquad t \in (t_2, t_3), \qquad t_2 = \cos^{-1}\frac{\xi - m}{r},$$

$$t_3 = \cos^{-1}\frac{-m}{r}$$

Exercise 1-2

length $= \int \frac{k}{\sin t}\sqrt{\alpha'^2 + q'^2}\, dt$

- $p := \underline{m + r}$, $q := \underline{m - r}$

- $$a = \frac{k}{2}\log(r^2 - m^2) = \frac{k}{2}\log(\underbrace{-pq}_{A}) = \frac{k}{2}\log A$$

$B + \frac{1}{B}$ · $$b = \frac{k}{2}\log\frac{-(p+1)(q+1)}{\underbrace{(p-1)(q-1)}_{B}} = \frac{k}{2}\log B$$

$C + \frac{1}{C}$ · $$c = \frac{k}{2}\log\frac{p(q^2-1)}{\underbrace{q(p^2-1)}_{C}} = \frac{k}{2}\log C$$

$$b = k\int_{t_1}^{\frac{\pi}{2}} \frac{dt}{\sin t} = \frac{k}{2}\log\frac{1-\cos t}{1+\cos t}\Big|_{t_1}^{\frac{\pi}{2}}$$

$\cos t_1 = \xi$ ....

Exercise 1-2

$A = \exp\frac{2a}{k}$ etc.

$$\left(A+\frac{1}{A}+2\right)\left(B+\frac{1}{B}+2\right)=4\left(C+\frac{1}{C}+2\right)$$

$$\left(\cosh\frac{2a}{k}+1\right)\left(\cosh\frac{2b}{k}+1\right)$$

$$= \cosh\frac{2c}{k}+1$$

$$\Rightarrow \boxed{\cosh\frac{a}{k}\cosh\frac{b}{k}=\cosh\frac{c}{k}}$$

Pythagorian formula

Exercise 1-2

$$ds^2 = k^2 \frac{dx^2 + dy^2}{y^2}$$

Pythagorian Theorem: $\to (k \to \infty)$: Euclidean metric

with appropriate coordinate change

$$\cosh \frac{a}{k} \cosh \frac{b}{k} = \cosh \frac{c}{k}.$$

($\to$ Euclidean pythagorian as $k \to \infty$)

$$\left(1 + \frac{1}{2}\frac{a^2}{k^2} + \cdots\right)\left(1 + \frac{1}{2}\frac{b^2}{k^2} + \cdots\right) = \cancel{1} + \frac{1}{2}\frac{c^2}{k^2} + \cdots$$

$$\| $$

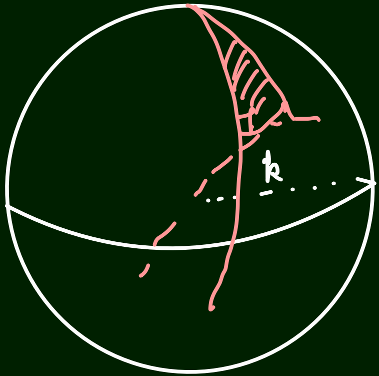
$$\cancel{1} + \frac{1}{2}\frac{a^2+b^2}{k^2} + \cdots$$

$2k^2$ $\qquad$ $2k^2$

$k \to \infty$ $\qquad$ $\boxed{a^2 + b^2 = c^2}$

Remark Spherical geometry.

a line : a great circle



$$\cos\frac{a}{R}\cos\frac{b}{R} = \cos\frac{c}{R}$$

$R \to \infty$

$$a^2 + b^2 = c^2$$

Problem Can one realize the non-Euclidean geometry as a surface in $\mathbb{R}^3$?

Answer

· locally yes (lect 2, 3, 7).

globally no (lect 4).

D. Hilbert (1911)

to construct surfaces of const. Gaussian curvature

· If $\mathbb{R}^3$ is replaced by some another space, yes (Lects 6 & 7)

global properties of solution of sine Gordon equation