Advanced Topics in Geometry B1 (MTH.B406)

Surfaces of constant Gaussian curvatrure

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Exercise 1-1

Show Lemma 1.12:

Lemma

Let $C_{c,r}$ *be a circle passing through* $\mathrm{P} = (x, y)$ *and* $\mathrm{Q} = (x + \Delta x, y + \Delta y)$*, where* $\Delta x \neq 0$*, and set*

$$(X, Y) := \psi_{2r} \circ \tau_{-c-r}(x, y),$$
$$(X + \Delta X, Y + \Delta Y) := \psi_{2r} \circ \tau_{-c-r}(x + \Delta x, y + \Delta y).$$

Then $\Delta X = 0$ *and*

$$\frac{\Delta Y}{Y} = \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{y} + o\left(\sqrt{(\Delta x)^2 + (\Delta y)^2}\right).$$

Exercise 1-1

$$(\xi, \eta) := \tau_{-c-r}(x, y) = (x - c - r, y)$$
$$\Rightarrow \qquad \tau_{-c-r}(x + \Delta x, y + \Delta y) = (\xi + \Delta x, \eta + \Delta y)$$

- Both (ξ, η), $(\xi + \Delta x, \eta + \Delta y)$ lie on $C_{-r,r}$.

$$\xi^2 + \eta^2 = -2r\xi, \qquad (\xi + \Delta x)^2 + (\eta + \Delta y)^2 = -2r(\xi + \Delta x).$$

$$\psi_{2r}(\xi, \eta) = \frac{4r^2}{\xi^2 + \eta^2}(\xi, \eta) = -2r\left(1, \frac{\eta}{\xi}\right) = (X, Y)$$
$$\psi_{2r}(\xi + \Delta x, \eta + \Delta y) = -2r\left(1, \frac{\eta + \Delta y}{\xi + \Delta x}\right) = (X + \Delta X, Y + \Delta Y).$$

Exercise 1-1

- $(\xi + \Delta x, \eta + \Delta y)$ lies on $C_{-r,r}$:

$$(\xi + \Delta x)^2 + (\eta + \Delta y)^2 = -2r(\xi + \Delta x)$$
$$\Rightarrow \qquad (\xi + r)\Delta x + \eta \Delta y = o(\sqrt{(\Delta x)^2 + (\Delta y)^2})$$

- (ξ, η) lies on $C_{-r,r}$:

$$r^2 = (\xi + r)^2 + \eta^2 = \left(\eta \frac{\Delta y}{\Delta x}\right)^2 + \eta^2 + o(1)$$
$$= \eta^2 \left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right) + o(1)$$

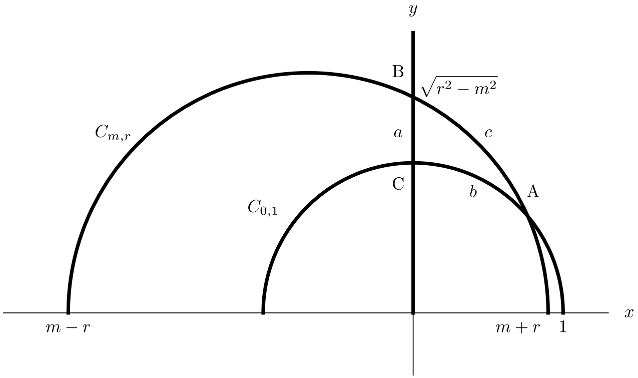
Exercise 1-1

$$
\begin{aligned}
\frac{\Delta Y}{Y} &= -2r\left(\frac{\eta+\Delta y}{\xi+\Delta x}-\frac{\eta}{\xi}\right)\Big/\left(-2r\frac{\eta}{\xi}\right)\\
&= \frac{\xi}{\eta}\left(\frac{1}{\xi}(\eta+\Delta y)\left(1-\frac{\Delta x}{\xi}\right)-\frac{\eta}{\xi}\right)+o(*)\\
&= \frac{\Delta y}{\eta}-\frac{\Delta x}{\xi}+o(*)\\
&= -\left(\frac{\xi+r}{\eta^2}+\frac{1}{\xi}\right)\Delta x+o(*)\\
&= -\frac{\xi(\xi+r)+\eta^2}{\xi\eta^2}+o(*)\\
&= \frac{1}{\eta}\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}+o(*).
\end{aligned}
$$

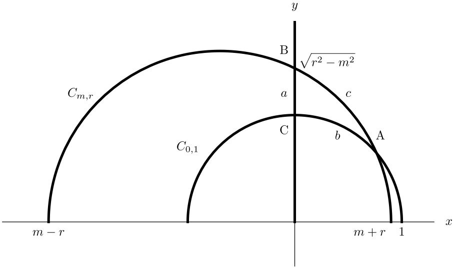
Exercise 1-2

Problem (Ex. 1-2)

Let A *be the intersection point of two lines* $C_{0,1}$ *and* $C_{m,r}$ $(r > 0$, $r^2 - m^2 > 1$, $r + m < 1)$, *and set* $\mathrm{B} = (0, \sqrt{r^2 - m^2})$, *and* $\mathrm{C} = (0, 1)$. *Find a relation of* $a := \mathrm{dist}(\mathrm{B}, \mathrm{C})$, $b := \mathrm{dist}(\mathrm{C}, \mathrm{A})$ *and* $c := \mathrm{dist}(\mathrm{A}, \mathrm{B})$.



Exercise 1-2



$$A = (\xi, \eta);\ \xi = \left(\frac{1+m^2-r^2}{2m}\right)$$

$$\begin{aligned}
&CB : (0, t) && t \in (1, \sqrt{r^2 - m^2}) && \\
&AC : (\cos t, \sin t) && t \in (t_1, \tfrac{\pi}{2}), && t_1 = \cos^{-1} \xi, \\
&AB : (r \cos t + m, r \sin t) && t \in (t_2, t_3), && t_2 = \cos^{-1} \frac{\xi - m}{r}, \\
& && && t_3 = \cos^{-1} \frac{-m}{r}
\end{aligned}$$

Exercise 1-2

$p := m + r$, $q := m - r$

$$\begin{aligned} a &= \frac{k}{2}\log(r^2 - m^2) = \frac{k}{2}\log(-pq) &&= \frac{k}{2}\log A \\ b &= \frac{k}{2}\log\frac{-(p+1)(q+1)}{(p-1)(q-1)} &&= \frac{k}{2}\log B \\ c &= \frac{k}{2}\log\frac{p(q^2-1)}{q(p^2-1)} &&= \frac{k}{2}\log C \end{aligned}$$

Exercise 1-2

$$\left(A + \frac{1}{A} + 2\right)\left(B + \frac{1}{B} + 2\right) = 4\left(C + \frac{1}{C} + 2\right)$$

Exercise 1-2

Pythagorian Theorem:

$$\cosh\frac{a}{k}\cosh\frac{b}{k}=\cosh\frac{c}{k}.$$