Advanced Topics in Geometry B1 (MTH.B406)

Surfaces of constant Gaussian curvatrure

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Immersed surfaces

$\subset \mathbb{R}^2$

$\{p_u, p_v\}$ linearly indep.

- $p\colon U \to \mathbb{R}^3$: a regular surface
- $\nu\colon U \to \mathbb{R}^3$: the unit normal vector field.

$$\nu = \frac{p_u \times p_v}{|p_u \times p_v|}$$

$\times$: the vector product (cross) in $\mathbb{R}^3$.

Fundammental forms

$p_u \cdot p_u$ $\qquad$ $p_u \cdot p_v = p_v \cdot p_u$

$$ds^2 := dp \cdot dp = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

$p_v \cdot p_v$

$$\widehat{I} := \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (p_u, p_v),$$

$$II := -d\nu \cdot dp = L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

$= -p_v \cdot \nu_v$

$$\widehat{II} := \begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (\nu_u, \nu_v)$$

$-p_u \cdot \nu_u$ $\qquad$ $-p_u \cdot \nu_v = -p_v \cdot \nu_u$

$= p_{uv} \cdot \nu$

$\underbrace{\det \widehat{I} > 0}$

$$A := \widehat{I}^{-1}\,\widehat{II} = \begin{pmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{pmatrix},$$

λ_1, λ_2 : the eigenvalues of A

$$\boxed{K := \lambda_1\lambda_2 = \det A = \frac{\det\,\widehat{II}}{\det\,\widehat{I}}}$$

Gaussian curvature.

$$H := \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}\operatorname{tr} A.$$

die Krümmung

Gauss-Bonnet Theorem; Geodesics

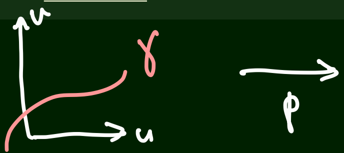
測地線

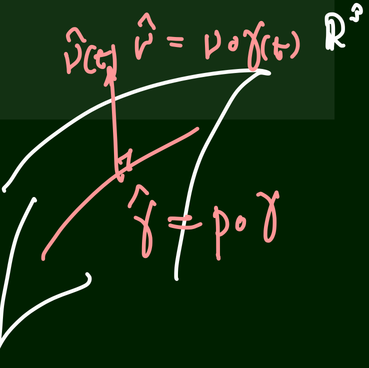
- $p\colon U \to \mathbb{R}^3$: a regular surface
- $\nu\colon U \to \mathbb{R}^3$: the unit normal vector field.
- $\gamma\colon I \to U$: a parametrized curve; $\hat{\gamma} = p \circ \gamma$, $\hat{\nu} = \nu \circ \gamma$.

Definition

γ (or $\hat{\gamma}$) is

- a <u>pregeodesic</u> if $\det(\hat{\gamma}', \hat{\gamma}'', \hat{\nu}) = 0$.
- a <u>geodesic</u> if $\hat{\gamma}'' \times \hat{\nu} = \mathbf{0}$.





Gauss-Bonnet Theorem; Geodesics

- $p\colon U \to \mathbb{R}^3$: a regular surface
- $\nu\colon U \to \mathbb{R}^3$: the unit normal vector field.
- $\gamma\colon I \to U$: a parametrized curve; $\hat\gamma = p \circ \gamma$, $\hat\nu = \nu \circ \gamma$.

· "straight line" of the surface.

Definition

γ (or $\hat\gamma$) is

- a <u>pregeodesic</u> if $\det(\hat\gamma', \hat\gamma'', \hat\nu) = 0$.
- a <u>geodesic</u> if $\hat\gamma'' \times \hat\nu = \mathbf{0}$.

a geodesic is a pregeodesic $(\because \hat\gamma'' /\!/ \hat\nu)$

a pregeodesic is a geodesic under an appropriate parameter change

Gauss-Bonnet Theorem for Geodesic Triangles

$dA = \sqrt{EG - F^2}\, du\, dv$ the area element.

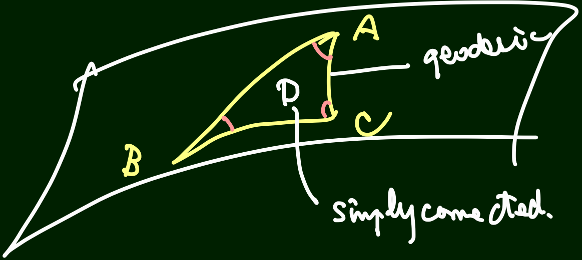
Theorem ([Theorem 10.6, UY17]).

K

Let $\triangle ABC$ be a geodesic triangle Then

$$\angle \mathrm{A} + \angle \mathrm{B} + \angle \mathrm{C} = \pi + \iint_{\underbrace{\triangle \mathrm{ABC}}_{D}} K \, d\mathcal{A},$$

where K and $d\mathcal{A}$ are the Gaussian curvature and the area element, respectively.



Lambert's theorem

Fact (Lambert (1728–1777)) (non positive)

In absolute geometry, there exists a negative constant K such that for all triangle ABC

$$\angle \mathrm{A} + \angle \mathrm{B} + \angle \mathrm{C} - \pi = K \triangle \mathrm{ABC}$$

where $\triangle \mathrm{ABC}$ denotes the area of the triangle.

Gauss-Bonnet theorem for constant K $(\underset{=}{<} 0)$.

$K = 0$: Euclidean geometry

$K < 0$: non-Euclidean geometry.

Q and A

Q: I don't understand Fact 1.2 (Lambert), what does "absolute geometry" mean? Is it synonymous with Euclidean geometry? Also how does the equality stated arise from the postulates I to IV?

No

- Euclidean geom. w/o. parallel postulate

Pseudospherical surfaces

Definition

A pseudospherical surface is a surface of constant Gaussian curvature -1.

note $\quad p \longmapsto cp \qquad c \neq 0$

$\Rightarrow K \longmapsto \frac{1}{c^2} K$

constant Gaussian curvature surfaces

$$\left.\begin{array}{l} K = 1 \\ K = 0 \\ K = -1 \end{array}\right\} \quad \text{up to homothety.}$$

pseudospherical surfaces ($K = -1$)

- local model of non-Euclidean geometry.

  * a simply connected p.s. surface $p : U \to \mathbb{R}^3$

    $\Rightarrow \quad \exists\ \varphi : U \to H$ an isometry

    H = upper half plane

    (s.t the pullback of ds^2 = first fundamental form, $\quad ds^2 = \dfrac{dx^2 + dy^2}{y^2}$) ($K = 1$)

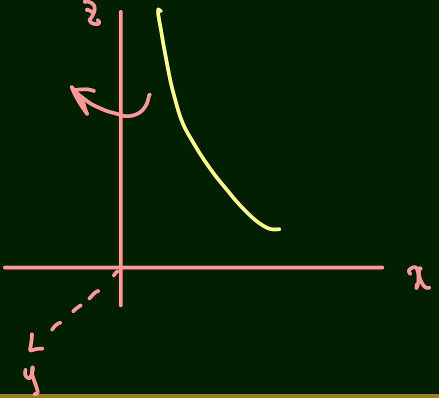
    (text 6 or 7)

Beltrami's pseudosphere 擬球面

$$p(u,v) := (\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v)$$

a surface of revolution w.r.to z-axis

and profile curve $(x(v), z(v)) = (\operatorname{sech} v, v - \tanh v)$



tractrix

(追跡線)

Beltrami's pseudosphere / Gaussian curvature

$$p(u,v) := (\operatorname{sech} v\cos u, \operatorname{sech} v\sin u, v-\tanh v)$$

$$\left.\begin{aligned} e_1(u) &:= (\cos u, \sin u, 0)\\ e_2(u) &:= (-\sin u, \cos u, 0)\\ e_3 &:= (0, 0, 1)\end{aligned}\right\}\ \text{orthonormal}$$

$(e_1)_u = e_2$

$(e_2)_u = -e_1$

$$p = \operatorname{sech} v\, e_1 + (v-\tanh v)\, e_3$$

$(v-\tanh v)' = 1-\operatorname{sech}^2 v = \tanh^2 v$

- $p_u = \operatorname{sech} v\, e_2$
- $p_v = -\operatorname{sech} v \tanh v\, e_1 + \tanh^2 v\, e_3$

  $= \tanh v(-\operatorname{sech} v\, e_1 + \tanh v\, e_3)$
- $\nu = \tanh v\, e_1 + \operatorname{sech} v\, e_3$

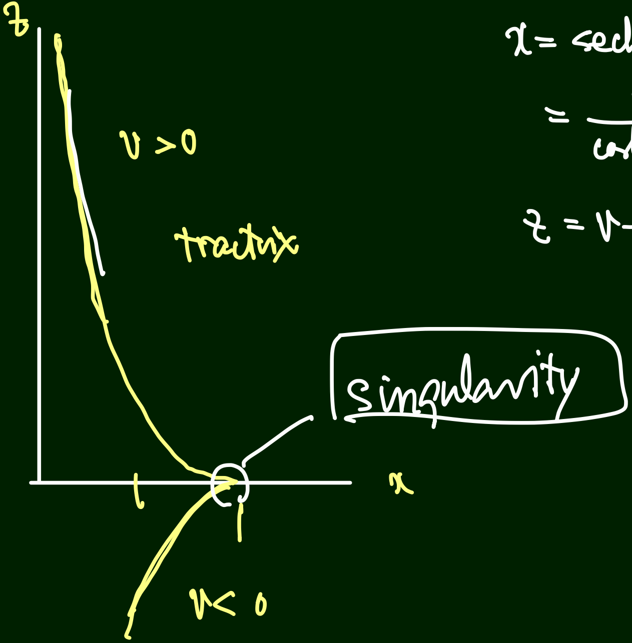
$$\nu = \tanh v\, e_1 + \operatorname{sech} v\, e_3$$
$$\nu_u = \tanh v\, e_2$$
$$\nu_v = \operatorname{sech}^2 v\, e_1 - \operatorname{sech} v \tanh v\, e_3$$

$$p_u \cdot p_u = \operatorname{sech}^2 v \;{}^{E}, \quad p_u \cdot p_v = 0 \;{}^{F}, \quad p_v \cdot p_v = \tanh^2 v \;{}^{G}$$
$$-p_u \cdot \nu_u = -\tanh v \operatorname{sech} v \;{}^{L}, \quad -p_u \cdot \nu_v = 0 \;{}^{M}$$
$$-p_v \cdot \nu_v = \tanh v \operatorname{sech} v \;\; N$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-\tanh^2 v \operatorname{sech}^2 v}{\operatorname{sech}^2 v \tanh^2 v}$$
$$= -1.$$

$$x = \operatorname{sech} v = \frac{1}{\cosh v}$$

$$z = v - \tanh v$$



Exercise 2-1

Hint: $x'x'' + z'z'' = 0$

Problem (Ex. 2-1)

Let $\gamma(t) = \big(x(t), z(t)\big)$ ($\gamma \in I$) be a parametrized curve on the xz-plane satisfying

$$\big(x'(t)\big)^2 + \big(z'(t)\big)^2 = 1 \qquad (t \in I),$$

unit speed

where $I \subset \mathbb{R}$ is an interval. Consider a surface

$$p_\gamma(s,t) := \big(x(t)\cos s, x(t)\sin s, z(t)\big),$$

which is a surface of revolution of prfile curve γ.

$x'' = x$

1. *Show that p_γ is pseudospherical if and only if $\cancel{z'' = z}$ holds.*
2. *Can one choose $I = \mathbb{R}$?*

$z' = \sqrt{1-(x')^2}$

Exercise 2-2

Problem (Ex. 2-2)

Let a and b be real numbers with $a \neq 0$. Compute the Gaussian curvature of the surface

$$p(u,v) = a(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v) + b(0,0,u).$$