Advanced Topics in Geometry B1 (MTH.B406)

Surfaces of constant Gaussian curvatrure

Kotaro Yamada
kotaro@math.sci.isct.ac.jp
http://www.official.kotaroy.com/class/2025/geom-b1

Institute of Science Tokyo

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Immersed surfaces

- $p\colon U \to \mathbb{R}^3$: a regular surface
- $\nu\colon U \to \mathbb{R}^3$: the unit normal vector field.

Fundammental forms

$$
\begin{aligned}
ds^2 &:= dp \cdot dp = E\,du^2 + 2F\,du\,dv + G\,dv^2,\\
\widehat{I} &:= \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (p_u, p_v),\\
II &:= -d\nu \cdot dp = L\,du^2 + 2M\,du\,dv + N\,dv^2,\\
\widehat{II} &:= \begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (\nu_u, \nu_v)
\end{aligned}
$$

Curvatures

$$A := \widehat{I}^{-1}\,\widehat{II} = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix}, \qquad \lambda_1, \lambda_2 \qquad \text{: the eigenvalues of } A$$

$$K := \lambda_1 \lambda_2 = \det A = \frac{\det \widehat{II}}{\det \widehat{I}}$$

$$H := \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}\operatorname{tr} A.$$

Gauss-Bonnet Theorem; Geodesics

- $p\colon U \to \mathbb{R}^3$: a regular surface
- $\nu\colon U \to \mathbb{R}^3$: the unit normal vector field.
- $\gamma\colon I \to U$: a parametrized curve; $\hat{\gamma} = p \circ \gamma$, $\hat{\nu} = \nu \circ \gamma$.

Definition

γ (or $\hat{\gamma}$) is

- a pregeodesic if $\det(\hat{\gamma}', \hat{\gamma}'', \hat{\nu}) = 0$.
- a geodesic if $\hat{\gamma}'' \times \hat{\nu} = \mathbf{0}$.

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Gauss-Bonnet Theorem for Geodesic Triangles

Theorem ([Theorem 10.6, UY17])

Let $\triangle ABC$ be a geodesic triangle Then

$$\angle \mathrm{A} + \angle \mathrm{B} + \angle \mathrm{C} = \pi + \iint_{\triangle \mathrm{ABC}} K \, d\mathcal{A},$$

where K and $d\mathcal{A}$ are the Gaussian curvature and the area element, respectively.

Lambert's theorem

Fact (Lambert (1728–1777))

In absolute geometry, there exists a <u>*negative*</u> *constant* K *such that for all triangle* ABC

$$\angle\mathrm{A} + \angle\mathrm{B} + \angle\mathrm{C} - \pi = K \triangle\mathrm{ABC}$$

where $\triangle\mathrm{ABC}$ *denotes the area of the triangle.*

Gauss-Bonnet theorem for constant K (< 0).

Q and A

Q: I don't understand Fact 1.2 (Lambert), what does "absolute geometry" mean? Is it synonymous with Euclidean geometry? Also how does the equality stated arise from the postulates I to IV?

Pseudospherical surfaces

Definition

A pseudospherical surface is a surface of constant Gaussian curvature -1.

Beltrami's pseudosphere

$$p(u, v) := \big(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v\big)$$

Exercise 2-1

Problem (Ex. 2-1)

Let $\gamma(t) = \big(x(t), z(t)\big)$ *(*$\gamma \in I$*) be a parametrized curve on the* xz*-plane satisfying*

$$\big(x'(t)\big)^2 + (z'(t))^2 = 1 \qquad (t \in I),$$

where $I \subset \mathbb{R}$ *is an interval. Consider a surface*

$$p_\gamma(s,t) := \big(x(t)\cos s, x(t)\sin s, z(t)\big),$$

which is a <u>*surface of revolution*</u> *of* <u>*prfile curve*</u> γ*.*

1. *Show that* p_γ *is pseudospherical if and only if* $z'' = z$ *holds.*
2. *Can one choose* $I = \mathbb{R}$*?*

Exercise 2-2

Problem (Ex. 2-2)

Let a and b be real numbers with $a \neq 0$. Compute the Gaussian curvature of the surface

$$p(u,v) = a(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v) + b(0,0,u).$$