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Kotaro Yamada
kotaro@math.sci.isct.ac.jp

Info. Sheet 2; Advanced Topics in Geometry B1 (MTH.B406)

Corrections

- Blackboard C, after page 7; compsition ⇒ composition
- Lecture Note, page 2, line 1 (just after Figure 2.): postulates ⇒ postulate
- Lecture Note, page 2, line 8: implies to V ⇒ implies V
- Lecture Note, page 2, line 19: Lobachevsky (1782–1856) ⇒ Lobachevsky (1792–1856)
- Lecture Note, page 2, line 19: Bolyai (1802–1827) ⇒ Bolyai (1802–1860)
- Lecture Note, page 3, line 7: $(x-a)^2+y^2\}$ ⇒ $(x-a)^2+y^2=r^2\}$.
- Lecture Note, page 3, line 8: centered at at $(c,0)$ ⇒ centered at $(a,0)$
- Lecture Note, page 3, Definition 1.5: **congruence** it maps ⇒ **congruence** if it maps
- Lecture Note, page 4, line 9: differntiability ⇒ differentiability
- Lecture Note, page 4, line 11: $\operatorname{dist}(ry, r(y+\Delta y))$ ⇒ $\operatorname{dist}(0, r(y+\Delta y))$
- Lecture Note, page 4, line 12: Remove "$=\operatorname{dist}(ry, r(y+\Delta y))$"
- Lecture Note, page 4, Lemma 1.12: $C_{a,r}$ ⇒ $C_{c,r}$
- Lecture Note, page 4, eq. (1.4):
$$\left(\frac{dy^2}{dt}\right) \quad \Rightarrow \left(\frac{dy}{dt}\right)^2$$
- Lecture Note, page 4, Exercise 1-2: a ⇒ m (4 times) except a in "$a := \operatorname{dist}(\mathrm{B},\mathrm{C})$"

Students' comments

- 練習問題が見た目より難しかったです．1-2 は解けなかったので解説を楽しみにします．
The exercises were harder than they looked. I couldn't solve 1-2, so I look forward to the explanation.

Lecturer's comment Sorry. I noticed that I should add several hints.

Q 1: Upper-half space model やユークリッドのモデルでは，世界となる集合の元が「点」を表しますが，集合の元以外のものが「点」となりえるモデルを考えることがありますか．
In the upper-half space model and Euclid's model, an element of the set which represents the "world" is regarded as a "point". Is there a situation which consider something "not an element of a set" as "a point"?

A: I think that a "world" is a set whose elements are considered to be points, usually.

Q 2: 構義（原文ママ：講義のことか）では congruence で不変ということから distance の公式を導いていましたが，歴史的にもそのような流れで距離が求められたのでしょうか．もしそうなら congruence はどのようにして思いついたのでしょうか．
In the lecture, the distance formula was derived from the fact it is invariant under congruence, but was distance obtained historically in this way? If so, how did you come up with congruence?

A: In The Elements, it is implicit, but it can be regarded as defining the distance because it is "invariant under congruence". In fact, Proposition 2 of the Book 1 of The Elements asserts that a line segment equal to a given line segment can be generated from another point. This seems to include the fact that the moved segment is equal to the original segment. (Of course, definition of a circle and postulate III are applied, but these include the fact that the line segment can be moved.) In the second half, it is natural to define congruence to be a map preserve the lines. Then the explicit expression of congruences is derived.

Q 3: 余弦定理や正弦定理の類似はどのような形か．
What are non-Euclidean analogue of cosine and sine laws?

A: We assume $k = 1$ in Lemma 1.10, and consider a triangle $\triangle\mathrm{ABC}$. Letting $a := \overline{\mathrm{BC}}$, $b := \overline{\mathrm{CA}}$ and $c := \overline{\mathrm{AB}}$ be lengths of the edges, and $A = \angle\mathrm{BAC}$, $B = \angle\mathrm{ABC}$, $C = \angle\mathrm{BCA}$. Then

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C \qquad \text{(cosine law)}$$

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c} \qquad \text{(sine law)}.$$

Q 4: 参考にした本では射影モデルが使われているが，上半平面もでるとはどのように関係しているか.
The book I referred used the projective model, but how does it relate to the upper half-plane as well?

A: The explicit relationship will be introduced in 7th Lecture.

Q 5: I don't understand Fact 1.2 (Lambert), what does "absolute geometry" mean? Is it synonymous with Euclidean geometry? Also how does the equality stated arise from the postulates I to IV?

A: See 2 lines above of Fact 1.2. In our context, the word "absolute geometry" means a plane geometry without the parallel postulate. The proof of Lambert's theorem is rather complicated and will not be treated in this lecture.