2 Surface of constant Gaussian Curvature

A quick review of surface theory

Immersed surfaces A C^∞-map $p\colon U \to \mathbb{R}^3$ defined on a domain $U \subset \mathbb{R}^2$ is called an *immersion* or a *parametrization of a regular surface* if

$$p_u(u,v) := \frac{\partial p}{\partial u}(u,v), \quad \text{and} \quad p_v(u,v) := \frac{\partial p}{\partial v}(u,v) \qquad \text{are linearly independent} \tag{2.1}$$

at each point $(u,v) \in U$. The *unit normal vector field* to an immersion $p\colon U \to \mathbb{R}^3$ is a C^∞-map $\nu\colon U \to \mathbb{R}^3$ satisfying

$$\nu \cdot p_u = \nu \cdot p_v = 0, \qquad |\nu| = 1 \tag{2.2}$$

for each point on U.

The *first fundamental form* ds^2 is defined by

$$ds^2 := dp \cdot dp = E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad \bigl(E := p_u \cdot p_u, F := p_u \cdot p_v = p_v \cdot p_u, G := p_v \cdot p_v\bigr), \tag{2.3}$$

where the subscript u (resp. v) means the partial derivative with respect to the variable u (resp. v). The three functions E, F and G defined on U are called the coefficients of the first fundamental form. On the other hand, the *second fundamental form* as

$$I\!I := -d\nu \cdot dp = L\,du^2 + 2M\,du\,dv + N\,dv^2, \qquad \bigl(L := -p_u \cdot \nu_u, M := -p_u \cdot \nu_v = -p_v \cdot \nu_u, N := -p_v \cdot \nu_v\bigr). \tag{2.4}$$

Here, we used a relation $\nu_u \cdot p_v = (\nu \cdot p_v)_u - \nu \cdot p_{vu} = 0 - \nu \cdot p_{vu} = -\nu \cdot p_{uv} = \nu_v \cdot p_u$. Define two symmetric matrices

$$\widehat{I} := \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (p_u, p_v), \qquad \widehat{I\!I} := \begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (\nu_u, \nu_v)$$

which are called the first and second fundamental matrices, respectively. Since $EG - F^2 = |p_u|^2|p_v|^2 - (p_u \cdot p_v)^2 > 0$, the first fundamental matrix $\widehat{I}$ is a regular matrix. The *area element* of the surface is defined as

$$d\mathcal{A} := \sqrt{EG - F^2}\,du\,dv. \tag{2.5}$$

Since $\widehat{I}$ is regular, the matrix

$$A := \widehat{I}^{-1}\,\widehat{I\!I} = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix}, \tag{2.6}$$

called the *Weingarten matrix*, is defined. The *Gaussian curvature* K and the *mean curvature* H are defined as

$$K := \lambda_1\lambda_2 = \det A = \frac{\det \widehat{I\!I}}{\det \widehat{I}}, \qquad H := \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}\operatorname{tr} A, \tag{2.7}$$

both of which do not depend on choice of parametrization.

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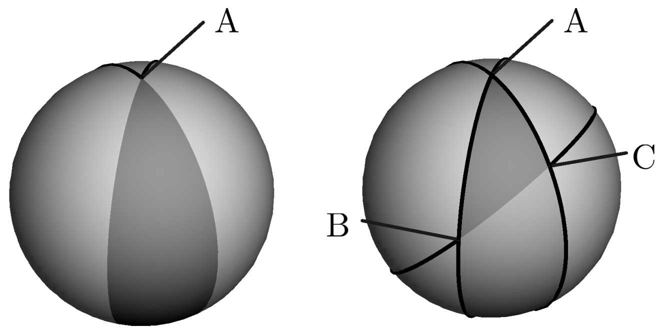

Figure 4: Gauss-Bonnet theorem for the sphere

Gauss-Bonnet theorem

Under the situation above, a parametrized curve $\gamma\colon I \to U$ (or its image $\hat\gamma = p\circ\gamma$ on the surface), where $I \subset \mathbb{R}$ is an interval, is called *pregeodesic* if it satisfies

$$\det\bigl(\hat\gamma'(t), \hat\gamma''(t), \hat\nu(t)\bigr) = 0 \qquad \left({}' = \frac{d}{dt}, \hat\gamma(t) = p\circ\gamma(t), \hat\nu(t) = p\circ\nu(t)\right) \tag{2.8}$$

for all $t \in I$. On the other hand, γ is called a *geodesic* if it satisfies

$$\hat\gamma''(t) \times \hat\nu(t) = 0, \tag{2.9}$$

where "$\times$" denotes the vector product of $\mathbb{R}^3$. In other words, the curve γ is a geodesic if and only if the acceleration vector $\hat\gamma''$ is proportional to the normal of the surface. The following is obvious.

Lemma 2.1. *A geodesic is a pregeodesic.*

Definition 2.2. A (geodesic) *triangle* on the surface is a closed domain of the surface which is homeomorphic to the closed disc, whose boundary consists of three segments AB, BC and CA of pregeodesics, which is called the *edge*. Three points A, B, C where two of the edges meet together are called *vertices* of the triangle. The *angle* of the triangle at the vertex A (resp. B, C) is the angle of tangent vectors of the geodesics CA and AB at A (resp. AB and BC at B, BC and CA at C).

Theorem 2.3 (Gauss-Bonnet theorem for triangles, [UY17, Theorem 10.6])**.** *Let $\triangle ABC$ be a geodesic triangle as in Definition 2.2. Then*

$$\angle\mathrm{A} + \angle\mathrm{B} + \angle\mathrm{C} = \pi + \iint_{\triangle\mathrm{ABC}} K\, d\mathcal{A},$$

where K and $d\mathcal{A}$ are the Gaussian curvature and the area element, respectively.

Example 2.4. Let S^2 be the unit sphere in $\mathbb{R}^3$. Then a pregeodesic is a great circle, that is, the intersection of a plane passing through the center of the sphere and S^2. Then for a geodesic triangle of S^2,

$$\angle\mathrm{A} + \angle\mathrm{B} + \angle\mathrm{C} = \pi + \text{the area of the surface}$$

holds because the Gaussian curvature of the surface is identically 1 (cf. Figure 4).

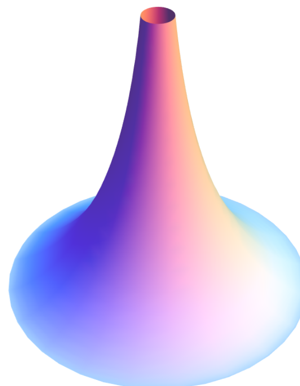

Figure 5: Beltrami's pseudosphere

Pseudospherical surfaces

Recall Lambet's result introduced in the previous section:

Fact 2.5 (Lambert (1728–1777))**.** *In absolute geometry, there exists a* <u>*negative*</u> *constant K such that for all triangle* ABC

$$\angle \mathrm{A} + \angle \mathrm{B} + \angle \mathrm{C} - \pi = K \triangle \mathrm{ABC}$$

where $\triangle$ABC *denotes the area of the triangle.*

Comparing this fact and Theorem 2.3, we notice that

> *A* surface of constant negative Gaussian curvature K satisfies the Lambert's theorem if we consider a geodesic as a "line".

In this seance, a surface of constant negative curvature can be regarded as a (local) *realization* of non-Euclidean geometry. The precise meaning of realization, and "local" will be clarified latter lectures. By a homothetic change $p \mapsto cp$, where c is a positive constant, the Gaussian curvature of the surface is changed as $K \mapsto c^{-2}K$. So when we consider a realization of non-Euclidean geometry, we may fix $K = -1$ without loss of generality.

Definition 2.6. A *pseudospherical surface* is a surface of constant Gaussian curvature -1.

Example 2.7 (The Pseudosphere)**.** A surface

$$p(u, v) := \bigl(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v\bigr) \qquad \bigl((u, v) \in (0, +\infty) \times (-\pi, \pi)\bigr)$$

is a pseudospherical surface, which is known as *Beltrami's pseudosphere* (Fig. 5).

Exercises

2-1 Let $\gamma(t) = \bigl(x(t), z(t)\bigr)$ $(\gamma \in I)$ be a parametrized curve on the xz-plane satisfying

$$(*) \qquad \bigl(x'(t)\bigr)^2 + (z'(t))^2 = 1 \qquad (t \in I),$$

where $I \subset \mathbb{R}$ is an interval. Consider a surface

$$p_\gamma(s,t) := \bigl(x(t)\cos s, x(t)\sin s, z(t)\bigr),$$

which is a *surface of revolution* of *profile curve* γ.

(1) Show that p_γ is pseudospherical if and only if $z'' = z$ holds.
(Hint: use the ralation $x'x'' + y'y'' = 0$ obtained by differentiating $(*)$.)

(2) Can one choose $I = \mathbb{R}$?

2-2 Let a and b be real numbers with $a \neq 0$. Compute the Gaussian curvature of the surface

$$p(u,v) = a(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v) + b(0,0,u).$$