Advanced Topics in Geometry B1 (MTH.B406)

Lecture Note

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Introduction

This is the second half of two series of lectures, *Advanced Topics in Geometry A1* and *B1*, in which the fundamental theorem for surface theory and its applications are treated.

Throughout this lecture, object of our interest is “surfaces in the Euclidean 3-space with constant negative Gaussian curvature”, which is a local model of non-Euclidean geometry.

In particular, we prove Hilbert’s theorem, which claims nonexistence of global model of non-Euclidean geometry as surfaces in the Euclidean 3-space. To prove the theorem, a way to construct constant negative Gaussian curvature surfaces via the fundamental theorem for surface theory is discussed.

Finally, we show that the global model of non-Euclidean geometry is realized as a surface in 3-dimensional Lorentz-Minkowski space.

An aim of the lectures for students is to observe mathematical view around undergraduate calculus, linear algebra, and highschool geometry.

13. June, 2025.

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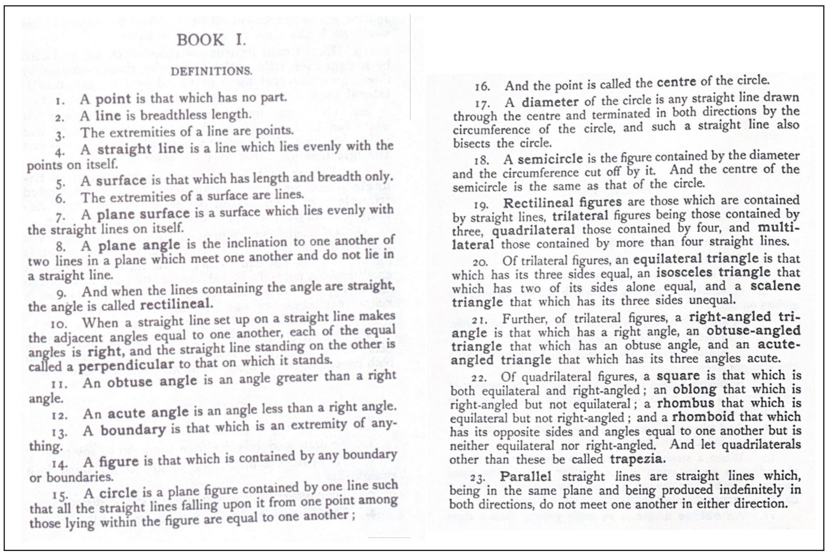
1 Non-Euclidean geometry

One of the oldest books on Mathematics is probably Euclid's *The Elements* (ca. 300 B. C.). Though the original book of this work has been lost, it became known throughout the world through manuscripts and translations into numerous languages [Euc56, Euc11].

The first book of *The Elements* begins with 23 *Definitions* (Figure 1), five *Postulates* (Figure 2, left), and *Common Notions* (fundamental laws concerning relations between quantities, Figure 2, right), on the basis of which the propositions of plane geometry are proved. This style of the book can be regarded as a prototype of the arguments in modern mathematics, and was studied mainly in Western Europe as "the norm of learning" until the modern era.



BOOK I.

DEFINITIONS.

1. A **point** is that which has no part.
2. A **line** is breadthless length.
3. The extremities of a line are points.
4. A **straight line** is a line which lies evenly with the points on itself.
5. A **surface** is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A **plane surface** is a surface which lies evenly with the straight lines on itself.
8. A **plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called **rectilineal.**
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right,** and the straight line standing on the other is called a **perpendicular** to that on which it stands.
11. An **obtuse angle** is an angle greater than a right angle.
12. An **acute angle** is an angle less than a right angle.
13. A **boundary** is that which is an extremity of anything.
14. A **figure** is that which is contained by any boundary or boundaries.
15. A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;
16. And the point is called the **centre** of the circle.
17. A **diameter** of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.
18. A **semicircle** is the figure contained by the diameter and the circumference cut off by it. And the centre of the semicircle is the same as that of the circle.
19. **Rectilineal figures** are those which are contained by straight lines, **trilateral** figures being those contained by three, **quadrilateral** those contained by four, and **multilateral** those contained by more than four straight lines.
20. Of trilateral figures, an **equilateral triangle** is that which has its three sides equal, an **isosceles triangle** that which has two of its sides alone equal, and a **scalene triangle** that which has its three sides unequal.
21. Further, of trilateral figures, a **right-angled triangle** is that which has a right angle, an **obtuse-angled triangle** that which has an obtuse angle, and an **acute-angled triangle** that which has its three angles acute.
22. Of quadrilateral figures, a **square** is that which is both equilateral and right-angled; an **oblong** that which is right-angled but not equilateral; a **rhombus** that which is equilateral but not right-angled; and a **rhomboid** that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called **trapezia.**
23. **Parallel** straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Figure 1: Definitions, Book 1 of the Elements [Euc56]

The establishment of *The Elements*, its meaning, the history of the *parallel postulate*, and the birth of *non-Euclidean geometry* will be treated in the subject *Transdisciplinary studies 20 : Mathematics Learn from History* (LAH.T420)[1].

Now, we state the five postulates of Euclid:

Postulate I. *To draw a straight line from any point to any point.*

This requires that there exists unique "line segment connecting two points".

Postulate II. *To produce a finite straight line continuously in a straight line.*

This requires that the line segment can be extended to any extent up to infinite length.

Postulate III. *To describe a circle with any centre and distance.*

Postulate IV. *That all right angles are equal to one another.*

Here, the definition of the right angles

> When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right* (Definition 10).

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[1] This course was completed in the first quarter of this academic year, but will be offered again next year.

POSTULATES.

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

COMMON NOTIONS.

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.

[7] 4. Things which coincide with one another are equal to one another.

[8] 5. The whole is greater than the part.

Figure 2: Postulates and Common Notions, Book 1 of the Elements [Euc56]

This postulate requires that the all right angles obtained by any pair of straight lines are equal. And the final postulate is known as the *parallel postulate*:

Postulate V. *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

This is a more complex statement than the postulates I–IV, and its geometric meaning is more difficult to understand immediately. The independence of the postulate V from the other postulates, i.e., whether or not I–IV implies V, remained as an open problem until the nineteenth century. Under the way of discussions on this problem, the following facts are obtained:

Fact 1.1. *The parallel postulate V is equivalent to that "there is the unique straight line passing through the given point and parallel to the given straight line".*

The geometry based on axiom system for Euclidean geometry without the parallel postulate is called the *absolute geometry*.

Fact 1.2 (Lambert (1728–1777))**.** *In absolute geometry, there exists a negative constant K such that for all triangle* ABC

$$\angle \mathrm{A} + \angle \mathrm{B} + \angle \mathrm{C} - \pi = K \triangle \mathrm{ABC}$$

where $\triangle\mathrm{ABC}$ *denotes the area of the triangle.*

The problem of the independence of the parallel postulate was settled at the end of the 19th century by Nikolai Lobachevsky (1792–1856), János Bolyai (1802–1860), or Carl Friedrich Gauss, in the following way: Assume the following denial of the parallel line axiom in addition to the axioms I–IV:

Postulate V'. *There are at least two straight lines passing through the given point and parallel to the given straight line.*

Then they show that the consistent geometric argument can be established. Such a geometry is called the *non-Euclidean geometry*.

However, this could be taken simply as a "play on words". In order for non-Euclidean geometry to be accepted as a mathematical theory, a "model" was necessary.

Before introducing the model of non-Euclidean geometry, let us review the model of Euclidean geometry. The world of plane geometry can be realized in the Cartesian plane $\mathbb{R}^2$. In particular, a point is an element of $\mathbb{R}^2$, and a straight line is a zero set$\{(x,y)\,;\, ax+by+c=0\}$ $(a,b,c \in \mathbb{R}; (a,b)\neq(0,0))$ of a linear function. Then, the world satisfying the five conventions is realized. In addition, the length of the line segments $\mathrm{A}=(x_1,y_1)$ and $\mathrm{B}=(x_2,y_2)$ connecting two points is given by

$$\overline{\mathrm{AB}} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2},$$

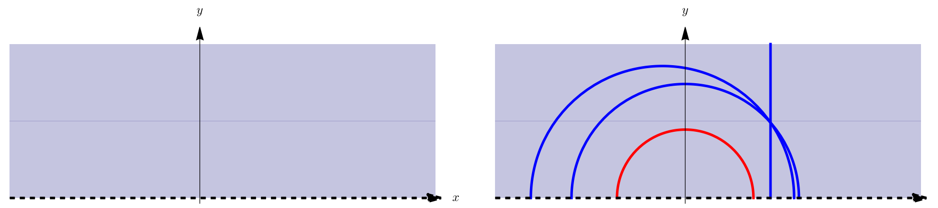

Figure 3: The upper-half plane

which is obtained by the *Pythagorean theorem* (Proposition 47 of *the Elements*). Thus, it seems important to obtain the expression for the lengths from the postulates.

A model of non-Euclidean geometry—Upper-half space model

The non-Euclidean geometry is realized by the following model: The world is the *upper-half plane*

$$H := \{(x, y) \in \mathbb{R}^2 \,;\, y > 0\} \tag{1.1}$$

and a *point* is an element of H. We think that a *line*, or a *straight line* on H is

$$\begin{gathered} C_{a,r} := \{(x, y) \in H \,;\, (x - a)^2 + y^2 = a^2\} \qquad (a \in \mathbb{R}, r > 0) \\ \text{(an upper-half of a circle centered at } (a, 0) \text{ on } x\text{-axis), or} \\ L_\lambda := \{(\lambda, y) \in H \,;\, y > 0\} \qquad (\lambda \in \mathbb{R}) \qquad \text{(a vertical half-line).} \end{gathered} \tag{1.2}$$

Definition 1.3 (a tentative definition)**.** The upper-half plane (1.1), in which curves as in (1.2) are considered as straight lines, is called the *hyperbolic plane*.

Let us observe that the hyperbolic plane suffices the postulates I–IV and V'. The following fact is obvious as seen in Figure 3 (a) and (b):

Lemma 1.4. *The hyperbolic plane suffices the postulate I and V'.*

To show postulates II and III are satisfied, we need to know how to measure "length", or "distance". To determine the distance of two points, we need some additional notions:

Definition 1.5 (A tentative definition of congruence)**.** A bijection $\varphi\colon H \to H$ is called a *congruence* if it maps an arbitrary line to a line.

By definition, it is obvious that a composition of congruences is also a congruence. The following is obvious:

Lemma 1.6. *For $\lambda \in \mathbb{R}$, a reflection φ_λ about a vertical line L_λ is congruence, where*

$$\varphi_\lambda\colon H \ni (x, y) \mapsto (2\lambda - x, y) \in H.$$

Corollary 1.7. *A horizontal translation $\tau_\lambda\colon (x, y) \to (x + \lambda, y)$ is a congruence.*

Proof. $\tau_\lambda = \varphi_0 \circ \varphi_{-\lambda/2}$. □

Lemma 1.8. *An inversion ψ_r about an (Euclidean) circle $C_{0,r}$ is a congruence, where*

$$\psi_r : H \ni (x, y) \mapsto \frac{r^2}{x^2 + y^2}(x, y) \in H$$

Corollary 1.9. *A dilation $\mu_r\colon (x, y) \mapsto (rx, ry)$ $(r > 0)$ is a congruence.*

Proof. $\mu_{a^4} = \psi_a \circ \psi_{1/a}$. $\square$

We denote

$$\operatorname{dist}(\mathrm{P}, \mathrm{Q}) = \text{the distance of P and Q},$$

and require that differentiability of dist (as a function of four variables), and

$$\operatorname{dist}(\mathrm{P}, \mathrm{Q}) = \operatorname{dist}(\varphi(\mathrm{P}), \varphi(\mathrm{Q})) \qquad \text{holds for each congruence } \varphi. \tag{1.3}$$

Lemma 1.10. *There exists a positive constant k such that*

$$\operatorname{dist}\bigl((x,y),(x,y+\Delta y)\bigr) = k\frac{\Delta y}{y} + o(\Delta y) \qquad (\Delta y \to 0).$$

Proof. Since a horizontal translation τ_s in Corollary 1.7 is a congruence, we may assume that $x = 0$. We set $d(y,\Delta y) := \operatorname{dist}\bigl((0,y),(0,y+\Delta y)\bigr)$. By differentiability, there exists a function $\delta(y)$ in y such that $d(y,\Delta y) = \delta(y)\Delta y + o(\Delta y)$. Hence by (1.3),

$$\begin{aligned} d(y,\Delta y) &= \operatorname{dist}\bigl((0,y),(0,y+\Delta y)\bigr) = \operatorname{dist}\bigl(\mu_r(0,y),\mu_r(0,y+\Delta y)\bigr) = \operatorname{dist}\left(0, r(y+\Delta y)\right) \\ &= \delta(ry)r\Delta y + o(\Delta y), \end{aligned}$$

and we have $y\delta(y) = ry\delta(ry)$. Since y and r are arbitrary positive numbers, there exists a constant k such that $\delta(y) = k/y$. $\square$

We fix k in Lemma 1.10 throughout this section.

Corollary 1.11. *The distance of* $\mathrm{P} = (x, y_1)$ *and* $\mathrm{Q} = (x, y_2)$ $(y_1 < y_2)$ *is* $k \log \frac{y_2}{y_1}$.

Proof. By Lemma 1.10,

$$\operatorname{dist}(\mathrm{P},\mathrm{Q}) = \int_{y_1}^{y_2} \frac{k}{y}\,dy = k\log\frac{y_2}{y_1} \qquad \square$$

Lemma 1.12. *Let $C_{c,r}$ be a circle as in* (1.2) *passing through* $\mathrm{P} = (x,y)$ *and* $\mathrm{Q} = (x+\Delta x, y+\Delta y)$, *where $\Delta x \neq 0$, and set*

$$(X,Y) := \psi_{2r}\circ\tau_{-c-r}(x,y), \qquad (X+\Delta X, Y+\Delta Y) := \psi_{2r}\circ\tau_{-c-r}(x+\Delta x, y+\Delta y).$$

Then $\Delta X = 0$ and

$$\frac{\Delta Y}{Y} = \frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{y} + o\left(\sqrt{(\Delta x)^2+(\Delta y)^2}\right).$$

Corollary 1.13. *Let $\gamma(t) := \bigl(x(t), y(t)\bigr)$ $(a < t < b)$ be a parametrized curve in H. Then the length of γ is given by*

$$\int_a^b \frac{k}{y}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_\gamma ds, \qquad \left(ds^2 := k^2\frac{dx^2+dy^2}{y^2}\right). \tag{1.4}$$

Exercises

1-1 Show Lemma 1.12.

1-2 Let A be the intersection point of two lines $C_{0,1}$ and $C_{m,r}$ ($r > 0$, $r^2 - m^2 > 1$, $r + m < 1$), and set $\mathrm{B} = (0, \sqrt{r^2-m^2})$, and $\mathrm{C} = (0,1)$. Find a relation of $a := \operatorname{dist}(\mathrm{B},\mathrm{C})$, $b := \operatorname{dist}(\mathrm{C},\mathrm{A})$ and $c := \operatorname{dist}(\mathrm{A},\mathrm{B})$, where $\operatorname{dist}(*,*)$ is given by a length of the line segment joining two points computed by (1.4).

2 Surface of constant Gaussian Curvature

A quick review of surface theory

Immersed surfaces A C^∞-map $p\colon U \to \mathbb{R}^3$ defined on a domain $U \subset \mathbb{R}^2$ is called an *immersion* or a *parametrization of a regular surface* if

$$(2.1)\qquad p_u(u,v) := \frac{\partial p}{\partial u}(u,v), \quad \text{and} \quad p_v(u,v) := \frac{\partial p}{\partial v}(u,v) \qquad \text{are linearly independent}$$

at each point $(u,v) \in U$. The *unit normal vector field* to an immersion $p\colon U \to \mathbb{R}^3$ is a C^∞-map $\nu\colon U \to \mathbb{R}^3$ satisfying

$$(2.2)\qquad \nu \cdot p_u = \nu \cdot p_v = 0, \qquad |\nu| = 1$$

for each point on U.

The *first fundamental form* ds^2 is defined by

$$(2.3)\quad ds^2 := dp \cdot dp = E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad \bigl(E := p_u \cdot p_u, F := p_u \cdot p_v = p_v \cdot p_u, G := p_v \cdot p_v\bigr),$$

where the subscript u (resp. v) means the partial derivative with respect to the variable u (resp. v). The three functions E, F and G defined on U are called the coefficients of the first fundamental form. On the other hand, the *second fundamental form* as

$$(2.4)\quad I\!I := -d\nu \cdot dp = L\,du^2 + 2M\,du\,dv + N\,dv^2, \qquad \bigl(L := -p_u \cdot \nu_u, M := -p_u \cdot \nu_v = -p_v \cdot \nu_u, N := -p_v \cdot \nu_v\bigr).$$

Here, we used a relation $\nu_u \cdot p_v = (\nu \cdot p_v)_u - \nu \cdot p_{vu} = 0 - \nu \cdot p_{vu} = -\nu \cdot p_{uv} = \nu_v \cdot p_u$. Define two symmetric matrices

$$\widehat{I} := \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (p_u, p_v), \qquad \widehat{I\!I} := \begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} p_u^T \\ p_v^T \end{pmatrix} (\nu_u, \nu_v)$$

which are called the first and second fundamental matrices, respectively. Since $EG - F^2 = |p_u|^2|p_v|^2 - (p_u \cdot p_v)^2 > 0$, the first fundamental matrix $\widehat{I}$ is a regular matrix. The *area element* of the surface is defined as

$$(2.5)\qquad d\mathcal{A} := \sqrt{EG - F^2}\,du\,dv.$$

Since $\widehat{I}$ is regular, the matrix

$$(2.6)\qquad A := \widehat{I}^{\,-1}\,\widehat{I\!I} = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix},$$

called the *Weingarten matrix*, is defined. The *Gaussian curvature* K and the *mean curvature* H are defined as

$$(2.7)\qquad K := \lambda_1\lambda_2 = \det A = \frac{\det \widehat{I\!I}}{\det \widehat{I}}, \qquad H := \frac{1}{2}(\lambda_1 + \lambda_2) = \frac{1}{2}\operatorname{tr} A,$$

both of which do not depend on the choice of parametrization.

20. June, 2025. Revised: 24. June, 2025)

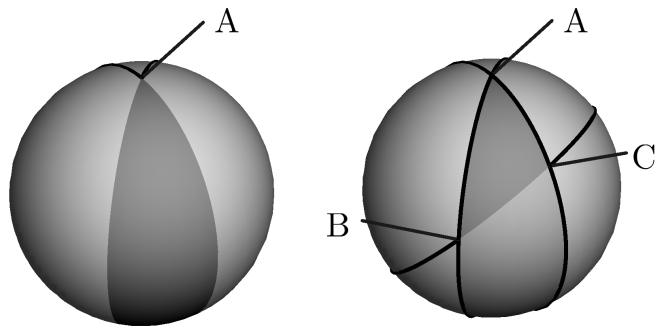

Figure 4: Gauss-Bonnet theorem for the sphere

Gauss-Bonnet theorem

Under the situation above, a parametrized curve $\gamma\colon I \to U$ (or its image $\hat{\gamma} = p \circ \gamma$ on the surface), where $I \subset \mathbb{R}$ is an interval, is called *pregeodesic* if it satisfies

$$\det\bigl(\hat{\gamma}'(t), \hat{\gamma}''(t), \hat{\nu}(t)\bigr) = 0 \qquad \left({}' = \frac{d}{dt}, \hat{\gamma}(t) = p \circ \gamma(t), \hat{\nu}(t) = \nu \circ \gamma(t)\right) \tag{2.8}$$

for all $t \in I$. On the other hand, γ is called a *geodesic* if it satisfies

$$\hat{\gamma}''(t) \times \hat{\nu}(t) = 0, \tag{2.9}$$

where "$\times$" denotes the vector product of $\mathbb{R}^3$. In other words, the curve γ is a geodesic if and only if the acceleration vector $\hat{\gamma}''$ is proportional to the normal of the surface. The following is obvious.

Lemma 2.1. *A geodesic is a pregeodesic.*

Definition 2.2. A (geodesic) *triangle* on the surface is a closed domain of the surface which is homeomorphic to the closed disc, whose boundary consists of three segments AB, BC and CA of pregeodesics, which is called the *edge*. Three points A, B, C where two of the edges meet together are called *vertices* of the triangle. The *angle* of the triangle at the vertex A (resp. B, C) is the angle of tangent vectors of the geodesics CA and AB at A (resp. AB and BC at B, BC and CA at C).

Theorem 2.3 (Gauss-Bonnet theorem for triangles, [UY17, Theorem 10.6])**.** *Let $\triangle ABC$ be a geodesic triangle as in Definition 2.2. Then*

$$\angle \mathrm{A} + \angle \mathrm{B} + \angle \mathrm{C} = \pi + \iint_{\triangle \mathrm{ABC}} K\, d\mathcal{A},$$

where K and $d\mathcal{A}$ are the Gaussian curvature and the area element, respectively.

Example 2.4. Let S^2 be the unit sphere in $\mathbb{R}^3$. Then a pregeodesic is a great circle, that is, the intersection of a plane passing through the center of the sphere and S^2. Then for a geodesic triangle of S^2,

$$\angle \mathrm{A} + \angle \mathrm{B} + \angle \mathrm{C} = \pi + \text{the area of the surface}$$

holds because the Gaussian curvature of the surface is identically 1 (cf. Figure 4).

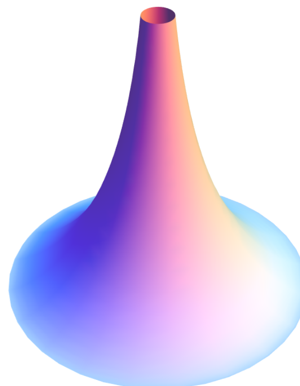

Figure 5: Beltrami's pseudosphere

Pseudospherical surfaces

Recall Lambet's result introduced in the previous section:

Fact 2.5 (Lambert (1728–1777))**.** *In absolute geometry, there exists a* <u>*negative*</u> *constant K such that for all triangle* ABC

$$\angle \mathrm{A} + \angle \mathrm{B} + \angle \mathrm{C} - \pi = K \triangle \mathrm{ABC}$$

where $\triangle$ABC *denotes the area of the triangle.*

Comparing this fact and Theorem 2.3, we notice that

> *A* surface of constant negative Gaussian curvature K satisfies the Lambert's theorem if we consider a geodesic as a "line".

In this sence, a surface of constant negative curvature can be regarded as a (local) *realization* of non-Euclidean geometry. The precise meaning of realization, and "local" will be clarified later lectures. By a homothetic change $p \mapsto cp$, where c is a positive constant, the Gaussian curvature of the surface is changed as $K \mapsto c^{-2}K$. So when we consider a realization of non-Euclidean geometry, we may fix $K = -1$ without loss of generality.

Definition 2.6. A *pseudospherical surface* is a surface of constant Gaussian curvature -1.

Example 2.7 (The Pseudosphere)**.** A surface

$$p(u,v) := \bigl(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v\bigr) \qquad \bigl((u,v) \in (-\pi,\pi) \times (0,+\infty)\bigr)$$

is a pseudospherical surface, which is known as *Beltrami's pseudosphere* (Fig. 5).

Exercises

2-1 Let $\gamma(t) = \bigl(x(t), z(t)\bigr)$ $(\gamma \in I)$ be a parametrized curve on the xz-plane satisfying

$$(*) \qquad \bigl(x'(t)\bigr)^2 + (z'(t))^2 = 1, \qquad x(t) > 0 \qquad (t \in I),$$

where $I \subset \mathbb{R}$ is an interval. Consider a surface

$$p_\gamma(s,t) := \bigl(x(t)\cos s, x(t)\sin s, z(t)\bigr),$$

which is a *surface of revolution* of *profile curve* γ.

(1) Show that p_γ is pseudospherical if and only if $x'' = x$ holds.
(Hint: use the ralation $x'x'' + z'z'' = 0$ obtained by differentiating $(*)$.)

(2) Can one choose $I = \mathbb{R}$?

2-2 Let a and b be real numbers with $a \neq 0$. Compute the Gaussian curvature of the surface

$$p(u,v) = a(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v) + b(0,0,u).$$

3 Pseudospherical surfaces and asymptotic Chebyshev net

Preliminaries

Let U and V be domains of $\mathbb{R}^n$

Definition 3.1. A C^∞ bijection $\varphi\colon V \to U$ is said to be a *diffeomorphism* if its inverse is also of class C^∞.

Lemma 3.2. *If $\varphi\colon V \to U$ is a diffeomorphism,*

$$(D\varphi)_{\varphi^{-1}(q)} \circ \bigl(D(\varphi^{-1})\bigr)_q = \mathrm{id}_{\mathbb{R}^n}, \qquad \textit{and} \qquad \bigl(D(\varphi^{-1})\bigr)_{\varphi(p)} \circ (D\varphi)_p = \mathrm{id}_{\mathbb{R}^n}$$

hold at each point of $q \in U$ and $p \in V$, where $D\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$ and $D(\varphi^{-1})\colon \mathbb{R}^m \to \mathbb{R}^n$ denote the differentials of the map φ and φ^{-1}. $(D\varphi)_p$ is a non-singular matrix on each point of $p \in V$.

Remark 3.3. Define $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $(x, y) = \varphi(\xi, \eta) = (\xi^3, \eta)$. Then the Jacobi matrix $D\varphi$ is computed as

$$D\varphi = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix} = \begin{pmatrix} 2\xi^2 & 0 \\ 0 & 1 \end{pmatrix}$$

which is singular at the origin. Hence φ is not a diffeomorphism though it is a bijection.

Theorem 3.4 (The inverse function theorem)**.** *Let $\varphi\colon U \to \mathbb{R}^n$ be a C^∞-map defined on a domain $U \subset \mathbb{R}^n$ and $p \in U$. Assume $(D\varphi)_p$ is non-singular. Then there exists a neighborhood $V \subset U$ of p such that $\varphi|_V\colon V \to \varphi(V)$ is a diffeomorphism. Moreover, $\bigl(D(\varphi^{-1})\bigr)_{\varphi(q)} = (D\varphi)_q^{-1}$ holds for each $q \in V$.*

Change of Parameters

Let $p\colon U \to \mathbb{R}^3$ be a regular parametrization of a surface in $\mathbb{R}^3$ and $\varphi\colon V \to U$ a diffeomorphism, where U and V are domains of $\mathbb{R}^2$. Then

$$\tilde{p} := p \circ \varphi\colon V \to \mathbb{R}^3 \tag{3.1}$$

gives another regular parametrization of a surface, whose image coincides with that of p. Such $\tilde{p}$ is said to be a parametrized surface obtained by the *coordinate change* φ of p.

Now we write $\varphi\colon (\xi, \eta) \to (u, v)$. Then by the chain rule, it holds that

$$(\tilde{p}_\xi, \tilde{p}_\eta) = (u_\xi p_u + v_\xi p_v, u_\eta p_u + v_\eta p_v) = (p_u, p_v)J, \qquad \text{where} \quad J := D\varphi = \begin{pmatrix} u_\xi & u_\eta \\ v_\xi & v_\eta \end{pmatrix}, \tag{3.2}$$

here p_u, p_v, $\tilde{p}_\xi$, $\tilde{p}_\eta$ are considered to be functions valued in the column-vectors.

We write the first fundamental form ds^2 (resp. $d\tilde{s}^2$) and the second fundamental form $I\!I$ (resp. $\widetilde{I\!I}$) of p (resp. $\tilde{p}$) as

$$\begin{aligned} ds^2 &= E\,du^2 + 2F\,du\,dv + G\,dv^2, &\qquad I\!I &= L\,du^2 + 2M\,du\,dv + N\,dv^2 \\ d\tilde{s}^2 &= \widetilde{E}\,d\xi^2 + 2\widetilde{F}\,d\xi\,d\eta + \widetilde{G}\,d\eta^2, &\qquad \widetilde{I\!I} &= \widetilde{L}\,d\xi^2 + 2\widetilde{M}\,d\xi\,d\eta + \widetilde{N}\,d\eta^2 \end{aligned}$$

Since the unit normal vector $\tilde{\nu}$ of $\tilde{p}$ coincides with $\nu \circ \varphi$, (3.2) yield

$$\begin{pmatrix} \widetilde{E} & \widetilde{F} \\ \widetilde{F} & \widetilde{G} \end{pmatrix} = J^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} J, \qquad \begin{pmatrix} \widetilde{L} & \widetilde{M} \\ \widetilde{M} & \widetilde{N} \end{pmatrix} = J^T \begin{pmatrix} L & M \\ M & N \end{pmatrix} J.$$

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This means that one obtain

$$ds^2 = d\tilde{s}^2, \qquad II = \widetilde{II}$$

by substituting

$$du = u_\xi\, d\xi + u_\eta\, d\eta, \qquad dv = v_\xi\, d\xi + v_\eta\, d\eta.$$

In other word, *the first and second fundamental forms are invariant under changes of parameters.* Moreover, the Gaussian curvature $K = (LN - M^2)/(EG - F^2)$ is also invariant under change of parameters.

Asymptotic parameters

For a surface of negative Gaussian curvature, there exists a parameter such that its second fundamental matrix is anti-diagonal, called an *asymptotic coordinate system.* In other words, a parameter (u, v) is an asymptotic coordinate system if and only if the second fundamental form is in the form

$$II = 2M\, du\, dv.$$

To prove this fact, we prepare

Lemma 3.5. *Let $\omega = \alpha\, du + \beta\, dv$ be a 1-form defined on a domain U of the uv-plane $\mathbb{R}^2$, where α and β are functions in (u, v). Assume $(\alpha, \beta) \neq (0, 0)$ at $\mathrm{P} \in U$. Then there exists a neighborhood $V \subset U$ of P and functions φ and ξ on V such that*

$$\varphi\omega = d\xi, \qquad \varphi(\mathrm{Q}) \neq 0 \quad \textit{for} \quad \mathrm{Q} \in V.$$

Proof. Let $\gamma(s) = \bigl(u_0(s), v_0(s)\bigr)$ a curve on U defined on an interval $I := (-\varepsilon, \varepsilon)$ $(\varepsilon > 0)$ satisfying $\gamma(0) = \mathrm{P}$, $\gamma'(s) \neq 0$ $(s \in I)$, and $\gamma'(0) = \bigl(u_0'(0), \dot{v}_0'(0)\bigr)$ satisfies

$$\alpha(\mathrm{P})u_0'(0) + \beta(\mathrm{P})v_0'(0) \neq 0. \tag{3.3}$$

Then for each $s \in I$, there exists a solution $\bigl((u^s(t), v^s(t)\bigr)$ $(t \in (-\delta_s, \delta_s))$ of a system of ordinary differential equations

$$\frac{d}{dt}u_s(t) = -\beta(u_s(t), v_s(t)), \quad \frac{d}{dt}v_s(t) = \alpha(u_s(t), v_s(t)), \quad u_s(0) = u(s), \quad v_s(0) = v(s).$$

Then, by a regularity of the solution of ordinary differential equations with respect to parameters, we obtain a smooth map

$$(s, t) \mapsto \bigl(u(s, t), v(s, t)\bigr) := \bigl(u_s(t), v_s(t)\bigr).$$

In particular,

$$\bigl(u(0,0), v(0,0)\bigr) = \mathrm{P}, \ \frac{\partial u}{\partial s}(0,0) = u_0'(0), \ \frac{\partial v}{\partial s}(0,0) = v_0'(0), \ \frac{\partial u}{\partial t}(0,0) = -\beta(\mathrm{P}), \ \frac{\partial v}{\partial t}(0,0) = \alpha(\mathrm{P})$$

hold. Hence by (3.3),

$$\det\begin{pmatrix} \frac{\partial u}{\partial s}(0,0) & \frac{\partial u}{\partial t}(0,0) \\ \frac{\partial v}{\partial s}(0,0) & \frac{\partial v}{\partial t}(0,0) \end{pmatrix} = \det\begin{pmatrix} u_0'(0) & -\beta(\mathrm{P}) \\ v_0'(0) & \alpha(\mathrm{P}) \end{pmatrix} \neq 0.$$

Thus, by the inverse function theorem, there exists a neighborhood V of P such that the map $(s, t) \mapsto (u, v)$ is a diffeomorphism, that is, (s, t) is a new coordinate system on $V \subset \mathbb{R}^2$. Using this parameter, we can write

$$\begin{aligned} \omega = \alpha\, du + \beta\, dv &= \alpha\left(\frac{\partial u}{\partial s}\, ds + \frac{\partial u}{\partial t}\, dt\right) + \beta\left(\frac{\partial v}{\partial s}\, ds + \frac{\partial v}{\partial t}\, dt\right) \\ &= \alpha(-\beta\, ds + u_t\, dt) + \beta(\alpha\, ds + v_t\, dt) = (u_t\alpha + v_t\beta)dt. \end{aligned}$$

So, by setting $\varphi := 1/(u_t\alpha + v_t\beta)$ and $\xi = t$, we have the conclusion. $\square$

Remark 3.6. Lemma 3.5 implies that any 1-form on a domain of $\mathbb{R}^2$ is locally a non-zero function multiple of an exact 1-form. The function φ in is called an *integrating factor* of the form ω.

Remark 3.7. Lemma 3.5 is the special (2-dimensional) case of *Caratheodory's principle*, which is often referred in the context of thermodynamics. In fact, Caratheodory's principle says that for any 1-form ω on n-manifold (or $\mathbb{R}^n$), there exists an integrating factor if and only if $\omega \wedge d\omega \neq 0$.

Proposition 3.8 (Asymptotic Coordinate system)**.** *Let $p\colon U \to \mathbb{R}^3$ be a regular parametrization of a surface in $\mathbb{R}^3$ whose Gaussian curvature is negative on U. Then for each $\mathrm{P} \in U$, there exists an asymptotic coordinate system on a neighborhood of* P*.*

Proof. Write the second fundamental form of p as $I\!I = L\,du^2 + 2M\,du\,dv + N\,dv^2$. Since the Gaussian curvature is negative, $-\kappa^2 := LN - M^2$ is negative. When $L(\mathrm{P}) = 0$, setting $u = \frac{1}{\sqrt{2}}(s-t)$, $v = \frac{1}{\sqrt{2}}(s+t)$ we get

$$I\!I(\mathrm{P}) = 2M(\mathrm{P})\,du\,dv = M(\mathrm{P})(ds-dt)(ds+dt) = M\,ds^2 - M\,dt^2.$$

Since $L(\mathrm{P}) = 0$, $\kappa(\mathrm{P})^2 = M^2(\mathrm{P}) \neq 0$, and hence the first coefficient of $I\!I$ with respect to the coordinate system (s,t) is not zero. Thus, we may assume $L \neq 0$ holds on a neighborhood of P, without loss of generality.

When $L \neq 0$,

$$\begin{aligned} I\!I &= L\left(du + \frac{M}{L}\,dv\right)^2 + \frac{LN-M^2}{L}dv^2 = L\left(\left(du + \frac{M}{L}\,dv\right)^2 - \left(\frac{\kappa}{L}\,dv\right)^2\right) \\ &= L\left(du + \frac{M+\kappa}{L}\,dv\right)\left(du + \frac{M-\kappa}{L}\,dv\right) \end{aligned}$$

Then by Lemma 3.5, there exists functions ξ, η, φ and ψ such that $\varphi(\mathrm{P}) \neq 0$, $\psi(\mathrm{P}) \neq 0$ and

$$du + \frac{M+\kappa}{L}\,dv = \varphi\,d\xi, \quad du + \frac{M-\kappa}{L}\,dv = \psi\,d\eta.$$

Here

$$\det\begin{pmatrix} \xi_u & \xi_v \\ \eta_u & \eta_v \end{pmatrix} = \frac{1}{\varphi\psi}\det\begin{pmatrix} 1 & \frac{M+\kappa}{L} \\ 1 & \frac{M-\kappa}{L} \end{pmatrix} = \frac{1}{\varphi\psi}\frac{2\kappa}{L} \neq 0$$

holds at P. Hence $(s,t) \mapsto (\xi,\eta)$ is a change of coordinates, and

$$I\!I = 2\widetilde{M}\,d\xi\,d\eta, \qquad (2\widetilde{M} = L\varphi\psi).$$

So (ξ,η) is an asymptotic coordinate system. □

Asymptotic Chebyshev net

Theorem 3.9. *For a each point* P *of a surface of constant negative Gaussian curvature $-k^2$, there exists a neighborhood U of* P *and coordinate system (ξ,η) such that the first and second fundamental forms are in the form*

$$ds^2 = d\xi^2 + 2\cos\theta\,d\xi\,d\eta + d\eta^2, \qquad I\!I = 2k\sin\theta\,d\xi\,d\eta, \tag{3.4}$$

where θ is a smooth function in (ξ,η) with $0 < \theta(\xi,\eta) < \pi$.

Proof. By Proposition 3.8, there exists an asymptotic coordinate system (u,v) around P:

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad I\!I = 2M\,du\,dv.$$

Then by the result in Exercise 5-1 of MTH.B405[2], $E_v = G_u = 0$ holds. Since both $E = p_u \cdot p_u$ and $G = p_v \cdot p_v$ are positive, we can write

$$E\,du^2 = \bigl(e(u)\,du\bigr)^2, \qquad G\,dv^2 = \bigl(g(v)\,dv\bigr)^2,$$

where $e(u)$ and $g(v)$ are positive functions in u and v, respectively. Set

$$\xi = \xi(u) = \int_{u_0}^{u} e(t)\,dt, \qquad \eta = \eta(v) = \int_{v_0}^{v} g(t)\,dt,$$

where $\mathrm{P} = (u_0, v_0)$. Then the map $(u, v) \mapsto (\xi(u), \eta(v))$ is a coordinate change because e and g are positive, and the first fundamental form and second fundamental form are written as

$$ds^2 = d\xi^2 + 2\widetilde{F}\,d\xi\,d\eta + d\,\eta^2, \quad II = 2\widetilde{M}\,d\xi\,d\eta.$$

Since the Gaussian curvature K is $-k^2$, we have

$$\widetilde{M}^2 = k^2\left(1 - \widetilde{F}^2\right), \qquad \text{that is,} \qquad \widetilde{F}^2 + \left(\frac{\widetilde{M}}{k}\right)^2 = 1.$$

So there exists a function θ such that

$$\widetilde{F} = \cos\theta, \qquad \widetilde{M} = k\sin\theta.$$

Since the surface is regular, $1 - \widetilde{F}^2 = 1 - \cos^2\theta > 0$ holds. So θ can move on the interval $(0, \pi)$ or $(\pi, 2\pi)$. In the latter case, replacing η by $-\eta$ and θ by $\pi - \theta$, we have the conclusion. $\square$

Remark 3.10. The parameter (ξ, η) as in (3.4) is called the *asymptotic Chebyshev net*.

Example 3.11. $p(u, v) := \bigl(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v\bigr)$.

Exercises

3-1 Let a and b be real numbers with $a \neq 0$ and

$$p(u, v) = a(\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v) + b(0, 0, u).$$

Find a coordinate change $(u, v) \mapsto (\xi, \eta)$ to an asymptotic Chebyshev net for p, and give an explicit expression of θ as a function in (ξ, η).

3-2 Let (ξ, η) be an asymptotic Chebyshev net (3.4) on a surface. Assume another parameter (x, y) is also an asymptotic Chebyshev net. Prove that (x, y) satisfies

$$(x, y) = (\pm\xi + x_0, \pm\eta + y_0) \qquad \text{or} \qquad (x, y) = (\pm\eta + x_0, \pm\xi + y_0)$$

where x_0 and y_0 are constants.

[2] Advanced Topics of Geometry A1

4 A construction of pseudospherical surfaces

4.1 Gauss-Weingarten equation

Let $p\colon U \to \mathbb{R}^3$ be a regular parametrization of a pseudospherical surface of constant Gaussian curvature -1 defined on a domain $U \subset \mathbb{R}^2$. By the result of the previous lecture, we may assume the coordinate system (x, y) on U is the asymptotic Chebyshev net:

$$ds^2 = dx^2 + 2\cos\theta\, dx\, dy + dy^2, \qquad I\!I = 2\sin\theta\, dx\, dy, \tag{4.1}$$

where $\theta = \theta(x, y)$ is a smooth function in (x, y) valued on an interval $(0, \pi)$. Now we define a new coordinate system (u, v) by

$$x = \frac{1}{2}(u - v), \qquad y = \frac{1}{2}(u + v), \tag{4.2}$$

and denote the new prametrization $p((u - v)/2, (u + v)/2)$ by $p(u, v)$. Then the first and second fundamental forms are written as

$$ds^2 = \cos^2\frac{\theta}{2}\, du^2 + \sin^2\frac{\theta}{2}\, dv^2, \qquad I\!I = \cos\frac{\theta}{2}\sin\frac{\theta}{2}(du^2 - dv^2). \tag{4.3}$$

Since $|p_u| = \cos\frac{\theta}{2}$, $|p_v| = \sin\frac{\theta}{2}$, and p_u is perpendicular to p_v, we can take the orthornomal frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ satisfying

$$p_u = \cos\frac{\theta}{2}\boldsymbol{e}_1, \quad p_v = \sin\frac{\theta}{2}\boldsymbol{e}_2, \quad \nu = \boldsymbol{e}_3, \tag{4.4}$$

where ν is the unit normal vector field of p. So we get the map

$$\mathcal{F} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)\colon U \longrightarrow \mathrm{SO}(3)$$

called the *frame* or an *adapted frame* of the surface, here SO(3) is the set of 3×3-orthogonal matrices with positive determinant. The following formula is a consequence of the Gauss-Weingarten equation (cf. Theorem 4.2 in MTH.B405, see also Exercise 4-2 in the same class).

Proposition 4.1. *Under the situation above, the frame $\mathcal{F}$ satisfies*

$$\begin{cases} \mathcal{F}_u &= \mathcal{F}\varOmega \\ \mathcal{F}_v &= \mathcal{F}\varLambda \end{cases}; \qquad \varOmega = \begin{pmatrix} 0 & -\theta_v/2 & -\sin\frac{\theta}{2} \\ \theta_v/2 & 0 & 0 \\ \sin\frac{\theta}{2} & 0 & 0 \end{pmatrix}, \quad \varLambda = \begin{pmatrix} 0 & -\theta_u/2 & 0 \\ \theta_u/2 & 0 & \cos\frac{\theta}{2} \\ 0 & -\cos\frac{\theta}{2} & 0 \end{pmatrix}.$$

Moreover, the function $\theta = \theta(u, v)$ satisfies the sine-Gordon equation

$$\theta_{uu} - \theta_{vv} = \sin\theta. \tag{4.5}$$

Proof. In spite of the direct conclusion of the Gauss-Weingarten equation, we'll give a direct proof for a sake of convenience. Differentiating the first equality of (4.4) in u, we have

$$p_{uu} = -\frac{\theta_u}{2}\sin\frac{\theta}{2}\boldsymbol{e}_1 + \cos\frac{\theta}{2}(\boldsymbol{e}_1)_u, \tag{4.6}$$

$$p_{uu} \cdot \boldsymbol{e}_2 = \cos\frac{\theta}{2}\big((\boldsymbol{e}_1)_u\big) \cdot \boldsymbol{e}_2, \tag{4.7}$$

$$p_{uu} \cdot \boldsymbol{e}_3 = \cos\frac{\theta}{2}\big((\boldsymbol{e}_1)_u\big) \cdot \boldsymbol{e}_3 \tag{4.8}$$

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where the third equality is nothing but the definition of the second fundamental form. On the other hand, by the definition of the first and second fundamental forms, we have

$$\sin\frac{\theta}{2}\,p_{uu}\cdot\boldsymbol{e}_2 = p_{uu}\cdot p_v = (p_u\cdot p_v)_u - p_u\cdot p_{uv} = -\frac{1}{2}(p_u\cdot p_u)_v = \frac{\theta_v}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}, \tag{4.9}$$

$$p_{uu}\cdot\boldsymbol{e}_3 = p_{uu}\cdot\nu = \cos\frac{\theta}{2}\sin\frac{\theta}{2}. \tag{4.10}$$

Since $(\boldsymbol{e}_1)_u\cdot\boldsymbol{e}_1 = \frac{1}{2}(\boldsymbol{e}_1\cdot\boldsymbol{e}_1)_u = 0$, we have

$$(\boldsymbol{e}_1)_u = \frac{\theta_v}{2}\boldsymbol{e}_2 + \sin\frac{\theta}{2}\boldsymbol{e}_3,$$

which proves the first column of Ω. On the other hand,

$$0 = p_{vu}\cdot\boldsymbol{e}_3 = \left(\sin\frac{\theta}{2}\boldsymbol{e}_2\right)_u\cdot\boldsymbol{e}_3 = \sin\frac{\theta}{2}(\boldsymbol{e}_2)_u\cdot\boldsymbol{e}_3,$$

proving the $(3,2)$-component of Ω. Since $\mathcal{F}$ is orthogonal, Ω is skew-symmetric. Thus we get the expression of Ω. The components of Λ are obtained in the similar way. $\square$

Remark 4.2. The equation (4.5) is equivalent to the equation

$$\theta_{xy} = \sin\theta, \tag{4.11}$$

which is the integrability condition with respect to the asymptotic Chebyshev net.

As a converse assertion, the fundamental theorem for surface theory deduces

Theorem 4.3. *Let $\theta\colon U\to(0,\pi)$ be a smooth function defined on a simply connected domain $U\subset\mathbb{R}^2$ satisfying the sine-Gordon equation* (4.5) *Then there exists a regular parametrization $p\colon U\to\mathbb{R}^3$ of a pseudospherical surface whose first and second fundamental forms are written as* (4.3).

Example

As an example of Theorem 4.3, we construct the surfaces of revolution (cf. Exercise 2-1).

Sine-Gordon equation and the equation of pendulum: We assume the function $\theta=\theta(u,v)$ depends only on the variable v: $\theta=\theta(v)$. Then the sine-Gordon equation turns to be

$$\ddot{\theta} = -\sin\theta \qquad \left(\dot{} = \frac{d}{dv}\right), \tag{4.12}$$

which is the equation of the motion of pendulums. In particular,

$$\left(\frac{\dot{\theta}}{2}\right)^2 + \sin^2\frac{\theta}{2} = E^2 \tag{4.13}$$

holds, where E is a non-negative constant. When $E=0$, $\sin(\theta/2)$ must be zero, which does not satisfy $\theta\in(0,\pi)$. On the other hand, when $\underline{E=1}$, the solution is written in an elementary function:

$$\theta = \theta_1 := 4\tan^{-1}\frac{e^v-1}{e^v+1} = 4\tan^{-1}\tanh v \tag{4.14}$$

Solving Gauss-Weingarten equation: In our case, the Gauss-Weingarten equation (Proposition 4.1) is rewritten as

$$\begin{cases}\mathcal{F}_u &= \mathcal{F}\varOmega\\ \mathcal{F}_v &= \mathcal{F}\varLambda\end{cases};\qquad \varOmega=\begin{pmatrix}0 & -\dot\theta/2 & -\sin\frac{\theta}{2}\\ \dot\theta/2 & 0 & 0\\ \sin\frac{\theta}{2} & 0 & 0\end{pmatrix},\quad \varLambda=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \cos\frac{\theta}{2}\\ 0 & -\cos\frac{\theta}{2} & 0\end{pmatrix}. \tag{4.15}$$

Let

$$c=c(v):=\frac{\dot\theta(v)}{2E},\qquad s=s(v):=\frac{1}{E}\sin\frac{\theta(v)}{2}. \tag{4.16}$$

Then by (4.13) and (4.12), it holds that

$$c^2+s^2=1,\qquad \dot c=-\cos\frac{\theta}{2}s,\qquad \dot s=\cos\frac{\theta}{2}c. \tag{4.17}$$

Using these, we set the orthogonal matrix $P=P(v)$ by

$$P:=\begin{pmatrix}1 & 0 & 0\\ 0 & c & -s\\ 0 & s & c\end{pmatrix}. \tag{4.18}$$

Note that the third column of P is the 0-eigenvector of $\varOmega$. Since

$$\widetilde{\varOmega}:=P^{-1}\varOmega P=P^T\varOmega P=E\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

and both $\varOmega$ and P are functions depending only on v, the first equation of (4.15) is reduced to

$$(\mathcal{F}P)_u=(\mathcal{F}P)\widetilde{\varOmega},$$

which can be solved as

$$\mathcal{F}P=F_0(v)R(u),\qquad R(u):=\begin{pmatrix}\cos Eu & -\sin Eu & 0\\ \sin Eu & \cos Eu & 0\\ 0 & 0 & 1\end{pmatrix},$$

where F_0 is an SO(3)-valued function in v. Substituting this into the second equation of (4.15),

$$\begin{aligned}\dot F_0 &= (\mathcal{F}PR^T)_v\mathcal{F}_vPR^T+\mathcal{F}\dot PR^T=\mathcal{F}\varLambda PR^T+\mathcal{F}\dot PR^T\\ &= F_0RP^T\varLambda PR^T+F_0RP^T\dot PR^T=F_0R\left(P^T\varLambda P+P^T\dot P\right)R^T=O\end{aligned}$$

holds because of (4.17) and the definition of $\varLambda$. Hence $F_0(v)$ is constant, and by choosing an appropriate initial condition, we obtain

$$\mathcal{F}=(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3)=R(u)P(v). \tag{4.19}$$

Hence we have

$$\boldsymbol{e}_1=\begin{pmatrix}\cos Eu\\ \sin Eu\\ 0\end{pmatrix}=\boldsymbol{u}_1,\qquad \boldsymbol{e}_2=c(v)\begin{pmatrix}-\sin Eu\\ \cos Eu\\ 0\end{pmatrix}+\begin{pmatrix}0\\ 0\\ s(v)\end{pmatrix}=\frac{\dot\theta}{2E}\boldsymbol{u}_2+\frac{1}{E}\sin\frac{\theta}{2}\boldsymbol{u}_3,$$

where $R=(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3)$. By (4.4), the corresponding surface $p=p(u,v)$ satisfies

$$dp=\cos\frac{\theta(v)}{2}\boldsymbol{v}_1(u)\,du+\frac{\dot\theta(v)}{2E}\sin\frac{\theta(v)}{2}\boldsymbol{v}_2(u)\,dv+\frac{1}{E}\sin\frac{\theta(v)}{2}\boldsymbol{v}_3\,dv. \tag{4.20}$$

Integrating this, we obtain

$$p=\frac{-2}{E}\cos\frac{\theta}{2}\boldsymbol{v}_2+\frac{1}{E}\boldsymbol{v}_3\int_{v_0}^{v}\sin\frac{\theta(t)}{2}\,dt,$$

which is a surface of revolution.

Exercises

4-1 The constant function $\theta(u,v)=0$ is a solution of the sine-Gordon equation (4.5) although it does not satisfy the condition $0<\theta<\pi$. In this case, explain what happens on the solution of (**??**) and resulting "surface" $p(u,v)$.

4-2 Let $\theta=\theta(x,y)$ be a solution of the sine-Gordon equation $\theta_{xy}=\sin\theta$. Assume a function φ satisfies

$$\left(\frac{\varphi-\theta}{2}\right)_x = a\sin\frac{\varphi+\theta}{2}, \qquad \left(\frac{\varphi+\theta}{2}\right)_y = \frac{1}{a}\sin\frac{\varphi-\theta}{2},$$

where a is a non-zero constant. Prove that φ is also a solution of the sine-Gordon equation.

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Glossary

absolute geometry 絶対幾何学，中立幾何学, 2
angle 角, 6
asymptotic coordinate system 漸近座標系, 10
axiom system 公理系, 2

Beltrami's pseudosphere ベルトラミの擬球面, 7
bijection 全単射, 9

Cartesian plane, デカルト平面，座標平面, 2
chain rule チェイン・ルール（合成関数の微分公式）, 9
circle 円, 1
congruence 合同変換, 3

diffeomorphism 微分同相, 9

edge 辺, 6
Elements 原論, 1

Gaussian curvature ガウス曲率, 5
geodesic 測地線, 6

homothetic 相似, 7
hyperbolic plane 双曲平面, 3

immersion はめ込み, 5
integrating factor 積分因子，積分因数, 11
interval 区間, 6

line segment 線分, 1

mean curvature 平均曲率, 5

non-Euclidean geometry 非ユークリッド幾何, 1
non-singular matrix 非特異行列，正則行列, 9

parallel postulate 平行線公準, 1
parametrization パラメータ表示, 5
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profile curve 母線, 8
pseudospherical surface 擬球面的曲面, 7
Pythagorean theorem ピタゴラスの定理, 3

right angle 直角, 1

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surface of revolution 回転面, 8

triangle 三角形, 6

unit normal vector 単位法ベクトル, 5
upper-half plane 上半平面, 3

vertex 頂点 (*pl.* vertices), 6